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Research Methodology

Unit I : Research Project – Difference between a dessertation and a thesis – Basic requirements of a research degree – Writing a proposal – Ethical considerations – Different components of a Research Project – Literature review – Methodology – Results / data – Conclusions – Bibliography - Appendices.
Chapter 5 : Sec : 5.1 – 5.13, **Chapter 6 :** Sec : 6.1 – 6.7, 6.8 (6.8.1 only), 6.9 (6.9.1 only), 6.11, 6.12 (6.12.1 only), 6.13 in Book 1

Unit II : Some Special Distributions : The Gamma and Chi – Square distribution – The normal distribution.
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Chapter 4 : Sec : 4.5 – 4.9 in Book 2.
Exercise Problems : Chapter 4 : 4.42, 4.43, 4.50, 4.51, 4.68 – 4.74, 4.83 – 4.93.

Unit V : Limiting distributions, Stochastic, Convergence – Limiting moment generating functions – The Central Limit Theorem – Some theorems on Limiting Distributions.
Chapter 5 : Sec : 5.1 – 5.5 in Book 2.
Exercise Problems : Chapter 5 : 5.1 – 5.3, 5.7, 5.8, 5.11 – 5.13, 5.15, 5.16, 5.20 – 5.27, 5.30 – 5.35.

Text Book : 1. **Writing up your University Assignments and Research Projects – A Practical handbook,** Neil Murray and Geraldine Hughes, McGraw Hill Open University Press.

2. **Introduction to Mathematical Statistics,** Fourth Edition, Robert V. Hogg and Allen T.Craig, Pearson Education Asia.

Unit I

• Understanding the research project and writing process What is a research project? • What's the difference between a dissertation and a thesis? • The basic requirements of a research degree • Deciding on a research topic • Choosing and using your degree • Writing a proposal • Adopting the correct mind set • Studying independently • Attending research seminars, conferences etc. • Understanding disciplinary differences • The upgrading process (PhDs only) • Familiarity with 'Codes of practice' / Rules and Regulations • Ethical considerations • The importance of finding your own 'voice'... and why it can be challenging • Getting down to writing

What is a research project?

In fih is guide, we have used the term 'research project' to refer to that component of a degree programme which requires you, the student, to successfully design, conduct and write up a piece of research will form either a dissertation or a thesis.

What is the difference between a dissertation and a thesis?

Students are often unclear about the difference between a dissertation and a thesis. For the purpose of this writing guide, there are no significant differences between the two in as much as both apply the same general principles of academic writing style and share similar principles of structure, organization and formatting. Where they differ is i their respective levels of detail: because a dissertation is normally one of a number of written requirements of a Bachelors or Master degree(B.A, B.Sc, M.A, M.SC, M.Ed. or M.Phil) it will typically be shorter in length and less detailed and far-reaching. In contrast, a thesis is the sole written requirement for the PhD degree and constitutes the final product of a lengthy period of research (normally 3 years or more); as such, it is expected to be considerably longer, more detailed, and more far-reaching than a dissertation. Further more, a defining characteristic of a thesis is its originality and the fact that it adds significantly to the existing body of knowledge in the field with which it is concerned. While a dissertation will also involve original work, there is less emphasis on this aspect, and research that replicates a previous study, for example - perhaps in a slightly different context or by employing a slightly different methodology - may well be acceptable.

Finally, a word about the M.Phil degree: the Master of Philosophy degree can be a taught degree or a research-based degree. If is a taught degree, the research project will form one of a number of written assignments that will need to be submitted and assessed before the degree is awarded. As such it will constitute a dissertation. In the case of a research-based M.Phil degree, there will be little or no course work and the research project will become the main focus of your attention and the main subject of assessment. In this case, it will constitute a thesis, reflecting as it does a PhD, if in a more truncated form.

Before looking in detail at the process of actually writing a research report, let's look briefly at a number of general principles and procedures that can help you understand better the whole research process and orientate you to the task ahead. An understanding of these will help smooth your journey over the coming months by increasing your overall awareness as well as the effectiveness and efficiency with which you work. Unfortunately, all too often students learn these principles and procedures the hard way, through experience, and as a result the process of conduction research and producing a dissertation or thesis becomes far more taxing and fraught with difficulties than it needs to be. Many of the following suggestions have to do with thinking ahead, working systematically, and using all the resources at your disposal.

The basic requirements of a research degree

What is originality?

As we have seen, one of the defining characteristics of a thesis- and to a lesser extent a dissertation - is its originality. In both cases it is a requirement that the research you report on is original. But what does this mean? Simply, that your research must add something new to the body of knowledge that already exists in the field in a way that no one else has done and in so doing push the barriers of our understanding of it. However, in the case of a dissertation, that contribution to knowledge will, in all probability, be less substantial than that of a thesis; nevertheless; both should seek to offer something new and original.

Here's how one university Handbook of Academic Regulations for Research Degrees puts it, 'The thesis shall form a distinct contribution to the knowledge of the subject and afford evidence of originality by the discovery of new facts and/or by the exercise of independent critical power'.

Students often ask the question, 'Just how original does the research need to be ?' Unfortunately, this is a bit like asking how long a piece of string is. There have undoubtedly been cases where research students have produced work that has completely upended their field and changed the way people think about it. Such change will often be the result of a broad and quite fundamental reassessment of the state of their discipline or key aspects of it, and it may well result ultimately in a complete paradigm shift. However, these cases are few and far between and it is far more common for research students to focus on one (often quite narrow) aspect of their field - we've all heard of the doctoral student who spent 15 years of their life studying the mating habits of an organism barely visible to the human eye! Of course, small doesn't necessarily mean insignificant, and perhaps this realization is key to answering this question: 'Original' means original, regardless of the reach of your research and the potential scale of its implications. More important than either reach or scale is significance. The examiners - as well as the wider audience (academic or otherwise) - need to feel that your research is significant or worthwhile in the sense that it 'contributes to the knowledge of the subject'. As the earlier quotation makes clear, this can be either through the discovery of new facts and/or the exercise of independent critical power; and, as we have seen, the contribution may consist of shedding light on one minute aspect of a very large field.

Other requirements

Apart from originality, there are other requirements you will need to meet if you are to bring your research to a successful conclusion. It is important you are aware of these requirements before embarking on your project and as such we have listed for you a typical set of such requirement written in rather formal language!

The thesis shall

• consist of the candidate's own account of his/her investigations, the greater proportion of which shall have been undertaken during the period of registration under supervision for the degree;

• be an integrated whole and present a coherent argument

• give a critical assessment of the relevant literature, describe the method of research and its findings, include discussion on those findings and indicate in what respects they appear to the candidate to advance the study of the subject; and, in doing so, demonstrate a deep and synoptic understanding of the field of study(the candidate being able to place the thesis in a wider context), objectivity and the capacity for judgements in complex situations and autonomous work in that field;

- be written in English and the literary presentation shall be satisfactory, although the candidate, with the support of the supervisor, may make application for a thesis in the field of modern languages and literatures only to be written in the language of study, to be considered on an exceptional basis by the Research Degrees Board of Examiners with advice sought from the Subject Area Board in the Humanities; in such cases the thesis shall include additionally a submission of between 10,000 and 20,000 words which shall be written in English and shall summarize the main arguments of the thesis;
- not exceed 100,000 words(inclusive of footnotes but exclusive of appendices and bibliography, the word limit not applying to editions of a text or texts);
- include a full bibliography and references;
- demonstrate research skills relevant to the thesis being presented;
- be of a standard to merit publication in whole or in part in a revised form (for example, as a monograph or as a number of articles in learned journals).

Note:

Although the above guidelines relate specifically to a thesis, they apply equally to a dissertation, with the exception of word length which is substantially less in the case of a dissertation-typically 10,000-20,000 words. The word limits for both theses and dissertation can vary from one institution to another and even between departments of the same institution.

Writing a proposal

Before commencing with your research, you will be asked to submit a proposal describing the nature of your project and the motivation for it, and giving an indication of how you intend to conduct it. This is normally no more than two sides of A4 in length and serves tow important purposes. First, it forces you, tge researcher, to clarify your own thinking. Often, it is only when we have to explain our thinking to others that we realize our ideas are only half-cooked, and if we are not clear either. Second, a proposal gives the department an opportunity to judge whether the project is viable and whether you as a researcher have thought it through it through adequately and are capable of bringing it to fruition. A proposal also allows them to decide whether they have a specialist in the department qualified and willing to supervise the project.

The format of a proposal is fairly standard and needs to contain the following elements:

• **Title and subject:** the department needs to see a working title of the project, one that is concise and gives a clear indication of its focus.

• **Statement of aims and objectives:** this section should explain what your research is designed to achieve, what problem it seeks to address, and the nature of the key constructs pertinent to solving that problem. For example, if your objective is to establish whether private schools are more successful than state schools, your main construct will be success. Those indicators of success which can be measured are called variables - examination results for instance - and you need to explain the role these will play in your study. The statement of aims and objectives only needs to be brief and should follow naturally form your discussion of the project's. Remember: state clearly and precisely what the exact focus of your research is by identifying and listing your main research questions and discussing briefly those variables that will shed light on the main construct(s) involved. As we shall see later, while variables can help you as a researcher gain insight into the key constructs underpinning your research question(s), they can also muddy the waters in certain cases and make life quite difficult. The solution lines in the careful and creative design of your methodology.

- **Formulating your hypotheses:** Next, in light of your objectives, you will need to formulate a set of hypotheses. These are simply statements - expressed as assertions - about the anticipated outcomes of your study, and as such they indicate the different ways that you, the researcher, expect the study to turn out. They are typically phrased as follows:

To meet these objectives, I will test the following hypotheses:

1. The number of $A/A^{\star}$ grades achieved at GCSE level will be consistently higher in private schools than in state schools.

2. The number of A grades achieved at A-Level will be consistently higher in private schools than state schools.

3. The proportion of students successfully gaining entry to a university of their choice will be higher for private schools than for state schools.

- **An indication of your methodology:** having contextualized your study and established your aims and objectives, you will need to explain how you plan to achieve those aims and objectives; in other words, what methods you plan to use. Different parts of your project may require different methods of data collection and analysis and so you will need to become familiar with these and explain which methods you have used for what purposes and why. Equally, you may wish to state why you have chosen not to use certain methods.

- **Excepted outcomes:** although it would be foolish to make absolute predictions about research that has yet to be undertaken, you may well have expectations about the eventual findings of the project. Spell these out, being careful to justify them and not to exaggerate them.
- **A time frame for completion:** spell out how long you except your project to take. This information is helpful to your potential supervisor for it indicates that you have planned out the project. It also has the advantage of forcing you to think ahead and organize your research, and of establishing milestones which can help motivate you and maintain momentum - even if you are ultimately unable to maintain the schedule.

Of course, as all researchers and supervisors know and expect, many of these details will change once your research begins and develops. What is important at this stage is the need to show yourself and your department/ supervisor that you have a viable project, along with the tools and know-how to conduct it effectively and bring it to a successful conclusion.

In the case of undergraduate research, there tends to be more flexibility and less formality over dissertation proposals, although the key elements should still be present, as should a clear statement of what it is the project regardless of the level at which you are conducting your research. Furthermore, it is increasingly common at undergraduate level as well as postgraduate level for students to be asked to present their proposed projects to their peers, research groups and so on, as a way of clarifying their proposed projects to their own thinking and benefiting from the feedback of their audience.

Ethical considerations

Conducting research frequently requires sensitivity to the effects of your research methods on those around you - particularly your subjects. This especially true of the science and social science disciplines , which often involve working with people or animals and collecting data through interviews , questionnaires or laboratory experiments. As a researcher, you need to consider the possible moral dimensions of what you are doing; whether your behaviour could be detrimental in any way to your subjects ; whether you are being honest and open with them and if not, whether your research really depends on a lack of openness and whether the potential benefits of the research justify it . Also consider whether your methodology and the motivation for it could be misconstrued and legal action be taken against you. If you think you might be skating on thin ice and feel that your research could lead to problems , take advice, and do so as quickly as possible don't wait until the project is well underway; try to anticipate possible difficulties at the design stage so that you don't waste months on a methodology that is undermined by ethical problems. Following are a few tips on how you can avoid such problems.

Make sure your subjects are well informed: explain your methods, the reasons behind the use to which you will put the information they give you . Apart form anything else ,people usually feel less intimidated and more inclined to participate- and to do so wholeheartedly if they understand what they are contributing to , how and why . Equally , if you're unable to share with them the object of the data collection exercise without defeating its purpose, then try to explain this. You can (a) often to share this information with them once the data has been collected and (b) agree not to use the data if ,at that point , the subject is uncomfortable with your doing so.

Ask your subjects permission: put simply , allow them the chance to decline the opportunity to contribute to your research by asking them very explicitly whether they are happy to take part . It's best to avoid cajoling those who are clearly uncomfortable with doing so as this could lead to trouble later. In cases where the information you are requesting is particularly sensitive , you may wish to formalize your subject's participation via a signed agreement in which you state the nature of the research and the data collection exercise and the willingness of the subject to take part . If , for purposes of the experiment and the integrity of the data collected , certain information needs to be withheld , then you might include a clause to that effect and saying that the subject has agreed to participate under those conditions.

protect your subject's privacy: it is usually unnecessary to refer to subject by name as the data itself , rather than who provides it, is often the key and the main focus of attention. However , if you do need to refer to particular subjects ; you should not use their real names ; use pseudonyms instead . This helps ensure that the subjects feel relaxed about providing information , especially when it's of a personal nature.

share your recorded data and the results of your research: it can be a good idea to show your committing them to paper . This not only gives them a final opportunity to confirm that they are happy for you to go ahead and use the data they have provided , it also allows you the opportunity to check your own understanding or interpretation of what was said. Similarly, offering to share the final results of your research with contributions (whether subjects of not) is part of what we might call research etiquette!

Be courteous: showing recorded data and the results of your research to contributions is a courtesy on your part as a researcher in payment for their time, effort and trust in helping make your project a success. But too is being punctual, organized and to the point so that you inconvenience them as little as possible while efficiently and effectively collecting the data you need. Be sensitive where necessary and avoid accidentally causing offence. And of course , always remember to thank them for their help; after all , your first attempt at data collection may prove to be flawed and you may need to return for more!

Always strive to be honest and objective: this is something that we will look at later on . For the moment, however let's just say that being honest and objective isn't as straightforward as you night think. It can be tempting to read into your data what you want to find there and it is therefore important to continually question your own integrity and objectivity. As with astrology and star signs, peoples are naturally disposed to interpret things in a way that fits with their experience, expectation and hopes.

Unit II

What are the different components of a research project

Title page:

The title page of a research report should not be numbered. The pages of all other preliminary sections, however , should be numbered using roman numerals, with the page immediately following the title page being numbered as ‘ii’. The pages of the main body of the text are normally numbered with Arabic numerals (1,2,3,.....)

The title of your report needs to indicate the nature and purpose of your research. It should be brief and to the point, and contain the key words or concepts underlying the work. Below are two examples of thesis titles:

The Effects of Hillslope-channel coupling on catchment
Hydrological Response in Mediterranean Areas

The politics of council housing decline:
Divergent responses in rural england in the 1980s

Following are two reduced - size examples of thesis title pages designed to illustrate two types of lay out traditionally used:

IMMIGRATION AND SOCIAL COHESION

IN CONTEMPORARY BRITAIN:

AN ASSESSMENT OF GOVERNMENT

STRATEGY FROM 1945-2000

Martina Lopez

2004

King's college London

University of London

A thesis submitted to the university of London

for the degree of Doctor of Philosophy

The copy right of this thesis rests with author and no quotation from it or information derived from it may be published of the author

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What Makes One Speaker 'Better' than Another ;

An inquiry into the judgement process in

Foreign Language oral proficiency Assessment

by

Neil L.Murray

A Dissertation submitted in partial fulfilment of the requirements

for the degree of

M.PHIL IN ENGLISH AND APPLIED LINGUISTICS

Research centre for English and Applied Linguistics

University of Cambridge

August 1992

Abstract

Your abstract should be a summary of the essential elements of your research project. It should serve as an overview , providing the reader with a good indication of what he or she will find in the pages that follow .This is important because the abstract is the

most read part of any research report, for it is frequently on the basis of the abstract that people decide whether or not the report is relevant to their own research (or other) interests and therefore worth reading .Typically , abstracts are between 250 and 300 words in length and should not go beyond one side of A4 .An abstract will normally include:

1) a statement of the main question or problem (i.e the purpose of the research);

2)the methods used to address it;

3) the results obtained;

4) the conclusions reached.

sometimes the author may also give a brief account of any

recommendations for future research made in the thesis and which derive from the research and its finding as documented in the thesis

Acknowledgements

The acknowledgements section is where you as the researcher and writer of the report thank those individuals and institutions that have assisted with or contributed to your research in some way . This may be through the provision of funding, facilities , services or data , or less directly via discussion and consultation , advice, motivation, and simply empathy and friend ship during what can be a challenging time in your academic career . The one person who will almost certainly feature in the acknowledgements is your supervisor ! It is considered a matter of recognize of courtesy to recognize these people and institutions and to spell their names correctly!

Look at the following sample 'Acknowledgements' page

This thesis would not have been possible without the generous support of the Rothermere foundation . In 1986 , I received the Rothermere foundation fellowship , which is awarded yearly to a graduate student of Memorial University . The fellowship permits the recipient to study at any institution in the United Kingdom , and has supported many distinguished scholars in the years since it was first instituted in 1956 by viscount Rothermere who was then the Chancellor of Memorial University.

At the time of my application, I was fortunate to come to the attention of Dr Deirdre Wilson , who agreed to act as my supervisor. In the years during which this research has wound its leisurely way

to a conclusion , she has provided guidance , support, understanding and professional and personal assistance of the most valuable kind. Through Dr Wilson I learned about relevance theory; and whatever contribution relevance theory may be represented by this thesis , I have discovered in the theory itself the key to questions raised by my experience as a student and teacher of literature - questions which had never been satisfactorily addressed before . For this alone I am immensely grateful.

I wish also to acknowledge my gratitude to the department of linguistics at university college london for the patience , courtesy, and support I have unfailingly encountered in the long course of completing this work.

To Dr Abbas and Mrs shomais Afnan, and to Ms sahba Akhavan, T owe a considerable debt. Their openhearted hospitality allowed me to return to the United Kingdom and complete the work on and the writing of this thesis.

Dr peter Baehr was kind enough to share his own work with me. for the opportunity to read ‘founders, classic, and the concept of a canon’ (Baehr and O’Brien 1994), and to discuss the connections between his research and my own, I am very grateful.

List of content

It is important that your list of contents is detailed and reflects accurately the structure of the research report. It should be arranged according to chapter or section numbers, incorporating all headings and sub-headings as they appear in the text , along with the page numbers on which they start. In order to indicate the status of different sections of the text , it is common practice to use a decimal numbering system:

Note: see part 3, Appendices ,for a sample table of contents .

Tables , figures and illustrations are normally numbered consecutively throughout the research report , and completely independent of the decimal numbering system used elsewhere . They will therefore follow a simple figure 1, figure 2, ... ,pattern, regardless of where they appear in the report. In the list of contents, however, it is important to indicate to indicate the page number on which each table , figure or illustration appears.

List of acronyms and abbreviations

It is quite common to find a list of acronyms and abbreviations at the start of a research report, usually following the list of contents.

Not surprisingly , researchers will typically draw on many written sources during the course of their projects and will consequently find it necessary to make reference to these in their writing . For the sake of convenience , rather than repeatedly writing out in full the names of source materials it is quicker and easier to refer to those materials using shortened forms -acronyms and abbreviations. Although the meaning of each acronym and abbreviation should be made clear after its first mention in the main text of the report, it is normal practice to provide a key to the meanings of these shortened forms in the first pages of the report. This allows for quick and easy reference on the part of the reader .Following is a sample list of Acronyms and Abbreviations taken from a real thesis.

List of Acronyms and Abbreviations

A.P.M.S.P -Australian Pacific Mail Steam Packet Company

B.Hist - Business History

B.R.P.M - Brazil and River Plate Mail

C.O - Colonial Office

C.G.T - Compagnie Generate Transatlantique

G.S.A.V - Compania Sud-America de Vapores

Ec.H.R	-Economic History Review
fo.(s)	- Folio(s)
F.O	- Foreign Office
GreenWich	- National Maritime Museum
H.A.H.R	-Hispanic -American Historical Review
I.A.E.A	-Inter -American Economic History
J.E.H	-Journal of Economic History
J.T.H	-Journal of Transport History
M.G.C	- Manchester Guardian Commercial
P.S.N	-Pacific Steam Navigation Company
U.C.L	- University College London
W.L.C.C	- West India Committee Circular

Introduction

The general principles underlying the writing of introductions were discussed in part 1 of this guide. Here we will look more specifically at what an introduction to a research report needs to achieve, and therefore what elements it will typically include. These are as follows :

1) The motivation for your research: you need to explain why you decided to embark on your research project. As we have seen,

your motivation could be an observation you have made directly during the course of your professional life , a ‘knowledge gap’ which you have noticed in the literature of your subject,or some other source of inspiration. The introduction , then, is that part of the report where you indicate the provenance of your research put it in perspective and set the scene for what is to come in the pages that follow.

2) the nature of the investigation: this is where you should define clearly the research questions you intend to address in your investigation , the key constructs underpinning them, the variables that will be influential in your investigation , and a statement of your hypothesis.

3)A brief description of how you approached your research

questions: this component should be a concise account of how you carried out your investigation . It should serve as a preface to the main methodology section and as such the level of detail included should not go beyond what is necessary to give the reader a broad but clear overview of the approach you adopted in addressing your research questions.

of course, these different elements are all interconnected . For example , as part of your explanation of the motivation for or background to your research , you will almost certainly describe your research questions and the way in which they emerged . In other words , it was because you identified a problem or discovered an important area of inquiry , as yet unaddressed, that you decided to conduct your research, in the hope of providing clarification and new insight.

Literature review

Why have a literature review

The literature review typically follows the introduction to your research report and its importance cannot be overestimated. It is where you present , in summary from , other work (books , articles, document etc.) the content of which relates in some way to your own research. The purpose of the review is :

1) To show where your study fits into the broader scheme of things; how it connects with the existing body of knowledge on the subject or on other related issues. In doing so, it also shows how your own research is original and promises to contribute to that pool of knowledge . In other words, along with the introduction , it helps to contextualize or position' your research by placing it within a broader framework. This also helps you to avoid reinventing the wheel by needlessly repeating the work (and mistakes) of others.

2) To help you locate information that may be relevant to your own research.

3) To increase and display your knowledge of the subjects - to the examiners in particular - and to convince them and your peers of the need , relevance and importance of your research and the suitability of the methodology you have adopted. Presenting what has been researched and written on a subject is one way of showing what needs to be done. It can do this by indicating the inadequacies of previous studies, by building on the finding of previous studies by taking them a step further, by highlighting an area of inquiry as yet unaddressed or unrecognized , or simply by taking a completely different approach to a subject or problem . In doing so, it shows the significance and value of your own research.

4)to identify seminal(key,influential) works in your area of study.

5)to identify methods, approaches and techniques that could be relevant to your own research.

6)to familiarize yourself with different and or opposing views and to demonstrate your ability to critique and evaluate the work of other scholars.

Organizing the literature search

To ensure that you are familiar with the relevant work of scholars in your field , you will need to do a literature search. This can seem a daunting task as there may be a very large body of published material . your supervisor will be able to offer advice on the best way to approach the task, but here are a few tips to help

1)conduct a search for a limited number of key books and journal articles on your topic published over the last few years.(remember that many journals are now available on-line)

2) As you read the articles , summarize the main points

3) Do not check only books and articles that are directly relevant to your own research. work that may seem a little peripheral to your own research topic can often include information that is very relevant or which triggers new ideas or directions of thought.

4)As you move backwards chronologically through the literature be sure to check out any sources widely cited by authors you have read and which appear relevant to your own research.

5)As your read, try to organize the literature according to its importance or relevance to your topic area.

Structuring the literature review

We have seen that the literature review is not simply a chronological list of previously published work. It plays an important role in creating a structure or framework that will allow you to display not only your knowledge of the relevant literature , but also your ability to summarize and critique the information and ideas it contains coherently .you can demonstrate this ability by:

1)grouping texts(articles,chapters,books etc.) according to the similarity of their ideas or arguments;

2)grouping studies that focus on similar phenomena or share similar methodologies;

3)commenting on the main ideas that feature in each group of texts or studies , rather than simply quoting or paraphrasing them;

4)comparing and contrasting the different studies, viewpoints, methodologies and so on , and identifying for the reader those which have the greatest bearing on your own research;

5) indicating which articles, ideas, methodologies and so on will form the basis of your investigations.

some of the most important citations are those referring to articles in refereed journals and you should include these in your literature review. you should be very cautious about using internet sources as these are not peer reviewed and therefore do not carry the same weight.

Although the dangers of over-quoting were highlighted in section 2.3 (plagiarism), the very nature of the literature review section of your dissertation or thesis means that you will inevitably devote much of this section to presenting and discussing the works of other scholars . Therefore, provided your literature review does not consist solely of a 'list' of citations of other scholarship , you will not be penalized for this.

The language cf critiquing

something you should consider when writing any section of your report is variety: you should try to use a range of vocabulary and grammatical structures in order to avoid monotony. Because you will probably be referring to numerous authors and viewpoints in your literature review, you will need to find different ways of introducing the authors you cite. some of these were discussed earlier in section 2.3, however, the use of the active and passive forms warrants special mention here.

A note on the active and passive forms

Look at the following two examples .Although they share the same information content , their structures and the effects they have on the reader are different:

peters(1992) discovered that ....(active)

It was discovered by peters (1992) that..........(passive)

In case of the active form, where the discoverers name is placed at the beginning of the sentences , the discoverer himself is given more prominence. This is useful when you wish to emphasize the discoverer more than or as well as his discovery . However , if you wish to give more emphasis to the discovery (as opposed to the discoverer) you may choose to use the passive tense. The active is used more widely than the passive , partly because it is easier to read and often creates a feeling of grater fluency . Remember , as a general rule, whatever is placed at the beginning of a sentence or clause is given greater prominence and therefore receives greater emphasis.

Methodology

What is it and why is it important

The methodology section of a research report describes how you conducted your study and the methods you used to collect and analyse the data. The term ‘methodology’ refers to the general approach taken to the research process , while ‘methods’ refers more specifically to the various ways in which data is collected and analysed .

Regardless of the field in which you are conducting your research, the overall aim of the methodology section is the same: to provide the reader with an overview of the methods employed so that a judgement can be made as to how appropriate they are given the objectives of the research, and how valid the data is that they have generated.

The following guidance notes are not intended to provide a comprehensive description and discussion of the various research methodologies , tools and techniques , but to alert you to a number of key issues you will need to consider in deciding and presenting your research methodology . You should discuss the details of individual methods and their suitability for your particular research with your tutor or supervisor .

The methodology you choose to use will serve as the underpinnings for your entire study, so your selection of the most suitable methodology is crucial. If you make bad choices at this early stage, they will have a ripple effect throughout your research, weakening its integrity and leading to questionable findings. Remember: your research is only as valid (and therefore valuable) as the methodology upon which it is based . Of course , there are many other factors that can affect the overall validity of your research - for example , how effectively you apply your methodology and how logical the deductions are that you make from your data ; nevertheless , a study that is sound at the conceptual level is of primary importance. Implementing a poorly conceived study is like building a house on sand rather than on a firm foundation : it will never be secure and will eventually fail and collapse, and all the time and effort put into constructing it will be wasted.

In this section ,then, you should present your methodology and rationale accurately and completely , but also as concisely as possible . you should also mention those methodological tools you considered but did not employ (particularly if they were used in related studies) and give the reason(s) why you decided not to use them your particular study.

Styles of presentation

The way in which you present your data will depend in part on whether that data is qualitative or quantitative . Quantitative data is usually presented using figures set out in the form of tables , graphs, charts and diagrams .When you present information in this way, you must of course make reference to it in your text , adding commentary to highlight and explain key aspects of the data.

A qualitative study may also present statistical data and employ graphs, charts and so on, but other types of data will likely also feature in such studies data which, for examples, record people's behaviour, attitudes, beliefs and opinions. This kind of data will often lend itself more to a fuller description written in normal prose, with figures being used to support and clarify points made in the text, as opposed to the text merely explaining the data presented in figures, such as in a quantitative study .Any such description needs to be accurate , succinct and coherent.

When making reference to a table , figure, chart or diagram ,the following expressions may be helpful:

The graph in figure 2 illustrates this trend.

As can be seen in the graph below(figure 8) , there was a clear correlation between ....and........

figure 3 highlights this growth in income over the past decade.

The results obtained are presented as a bar chart in figure 15. They clearly indicate.......

The table in figure 4 records.......

The chart in figure 7 indicates suggests.........

The response times of subjects were recorded and plotted on a graph(figure 8) over 70 of respondents showed greater improvement in health as a result of taking the drug on a regular basis , as indicated in figure 24.

As figure 5 illustrates , observations over a 3- month period reinforced these initial perceptions.

Subjects' responses to the questionnaire were carefully compiled and recorded in tablature from (figure 16)

Conclusions

Although this section will have much the same form as any other conclusion (see section 2.2) it will differ in some ways and will

typically the following three closely connected elements:

1) A discussion of those inferences that can be drawn from your research: any inferences you make must be supported by the evidence you have provided in previous sections through rational argument and or the analysis data.

2)A statement of the contribution your research has made to the field of inquiry: the key requirement for a thesis is that it adds to the body of knowledge in a particular field by contributing something original. This section is therefore especially important , for in it you will be summarizing the contributing your own research has made, and it is essentially on that basis that it will be judged by the examiners and other scholars who read it .

3)Suggestions for future research: the most common way to end a dissertation or thesis is to suggest new avenues of investigations based on your own research as documented in your report. In other words, this is where you indicate how future research might build upon your own methods of investigation and the findings they have produced. part of this may involve highlighting problems that you had with your own approach and, based on your experience, suggesting alternatives to avoid similar such problems recurring.

Remember, no new information should appear in a conclusion, only inferences drawn from information that has already been presented elsewhere in the dissertation or thesis .Avoid unnecessary digressions and do not introduced new arguments.

Keep your concluding statements concise and to the point, present them in a logical order , and make sure they relate back to your research question(s).

Bibliography

A Bibliography is a complete list of references to the works you have consulted during the course of your research.A comprehensive and well laid out bibliography will be an important factor in how positively your work is evaluated by your peers, examiners etc. A good bibliography will

1)indicate that you have consulted others work and are aware of the debate arguments and practices in your field ,particularly as they relate to the subject of your own research.

2) add weight and credibility to your statements;

3)enable others to check the accuracy of your information and interpretations;

4)direct others to works you have found useful and to related publications;

5)acknowledge other peoples work and ideas

6)enable you and your readers to review the sources of your information ;

7)show that you are familiar with academic formatting conventions.

How to format your references

It is important to repeat that there are a variety of styles that are used for formatting references . We noted , for example, in section 2.3 that the Vancouver references system- also called the author-number ' system -is often employed in the science disciplines and used a number series indicate reference in the body of the text.

These references are then listed in the bibliography in the same numerical order as they appear in the body of the text .Look at the following examples:

1) Shepherd G.Neurobiology .Oxford University press; 1994.

2)Folstein M, Gilman S.Neurobiology of primary dementia.American psychiatric press,1998.

3)Grabe,L.Depression Reassessed. in : p.Davies , S.Mycroft,S.Dixon (eds).New psychological perspectives .Tamley press; 2007 p.57-70

4)Geck MJ ,Yoo S, Wang JC.Assessment of cervical ligamentous injury in trauma patients using MRI .J Spinal Disord. 2001;14(5):371-7

5)Morse SS.Factors in the emergence of infectious disease.Emerg Infect Dis[serial online] 1995 Jan- Mar [cited 1996 Jue 5];1(1):[24 screens] . Available from : URL:http:// WWW.cdc .gov/ nci-doc/EID/eid .htm

Be sure to check which style(s) are considered acceptable in your particular field. Once your have opted for a particular style, apply that style consistently and do not jump from one style to another. We shall focus in detail on the Harvard system.

Articles in journals:

peters ,M.(1992) performance and Accountability in post -industrial society.the crisis of British Universities .studies in Higher Education 17(2) 123-40

The first number (17) is the volume number and the second number (2) is the part number (where available) .The final numbers (123-40) are the page numbers of the article.

The conventions for joint (two) or multiple (more than two) authorship of articles are same as those used for books.

Diagrams and illustrations:

Klein F.(2006) Demographic trends in the indian subcontinent since 1990 in : V.Steppenwolf and M.Khalid (eds) The indian subcontinent :Human Geographical perspectives London : Cyclone press.

Electronic sources:

1)the authors name(if known);

2)the full title of the document ;

3)the www home page (if available);

4)the authors email address (if available);

5)the date of publication;

6)pathway directions for accessing the document;

7)the date you accessed the information.

Example:

Citing in the bibliography:

Simons,peter (2001) Audience participation . Theatre Reviews. http://WWW.big brother .terracom/frames-news .html(15 oct 2001)

Remember:

1)Every publication listed in the bibliography should have been cited in your thesis or used in its preparation .

2)Each publication should include the following elements in the order they are presented here:

The following examples show how referencing conventions are applied according to their sources:

Books:

personal authors:

clark, A.(2000)organisations ,competition and the business environment London. pearson.

brown ,G.and Atkins, W.(1990) effective teaching teaching. London .Routledge .

Coffield ,F,Borrill ,c. and Marshall , s.(1986) Growing up at the margins :young adults in the north east .Milton keynes :open university press

peters G.(1990, 2nd edn) real -time processing .London:Routledge.

where a book has more than one edition , be sure to state which edition of the book you have used.

Edited volumes:

Day , R.editor (1986) talking to learn :conversation in second language acquisition Rowley: Newbury House.

Organization as author:

British Medical Association .(1993) complementary medicine :the BMA guide to good practice. Oxford :Oxford University press.

Chapter in a book:

Pilfer ,M.(1994) Quality assurance in higher education in B.Wilkins(ed) issues in higher education .London : Falmer press , pp.77-92.

Major, R. (1987) 'A model for interlanguage phonology', in G.Ioup and S. Weinberger(eds) Interlanguage Phonology .Cambridge :Newbury house.

Theses or dissertations:

Murray ,N.(1996) communicative language teaching and Language Teacher Education. ph.D .thesis .University of london.

Official publication:

Department of health (1998) 1996 report of the committees on Toxicity ,Mutagenicity , Carcinogenicity of chemicals in food , consumer products and the environment. London :HMSO

1)author(s) surname(s) plus initial(s);

2)date of publication;

3)title of book (or title of book in which the work appears if it is a chapter);

4)title of journal(if a journal article);

5)volume/edition/ page numbers (if a journal article)

6)place of publication and name of publisher(if a book).

* All sources cited in your thesis should be listed alphabetically in the bibliography by author/organisation .

*If more than one book has been written by the same author (one or more as a single authorship and others in collaboration with other authors) the order in which they should be listed can be seen below. Note that for this order to apply , the single author appearing in category a) must also be the main author (i.e. that listed first) in those subsequent publications appearing in categories b) and c):

a) single -authored items are listed first

b) joint authored items are listed second

c) multiple authored items are listed last.

with in each of these 3 categories (a,band c) items should be listed in order of date, with the earlier-published items appearing first.

* If two or more items in the bibliography have the same year of publication, they should be listed with a small (lower -case) letter(a,b,c ,etc.) following the date. This convention should be followed both within the main body of the work and in the bibliography:

Field,A.(1985a)refers to this as.........

or (in the bibliography)

Field,A .(1985a) issues in language teaching .education 165(25)p.560

Field , A.(1985b) perspectives in the primary curriculum. cambridge journal of education 15(1) pp.41-9

Bibliography Management Software

Today there are a number of bibliography software packages available that make the process of creating a bibliography much easier and less time consuming by automatically generating and formatting references listed for you .Three such packages are refworks(WWW.refworks .com , a web-based service),references manager (see WWW.refman.com) and endnote (see WWW.endnote .com). often,universities will have licensing agreements with the provides of these packages allowing you to download them free to your computer. Alternatively, you can purchase them privately .Bibliographic management software allows you to create a database of references for books , journal articles , book chapters , dissertations , art work, recording, web pages and so on . These records can be entered manually or important directly from library catalogues and commercial databases. The software then uses this database to create and format a bibliography in a specific style while working in word processing software such as Micro soft word.

UNIT -III

Gamma Distribution

Gamma Function:

$\Gamma_{(\alpha)} = \int_0^\infty y^{\alpha-1}e^{-y}dy \longrightarrow (*)$ is called the Gamma Function for $\alpha > 0$.

If x=1 ,$\Gamma_{(1)} = \int_0^\infty e^{-y}dy = 1.$

Let $y = \frac{x}{\beta}$

$\frac{dy}{dx} = \frac{1}{\beta}$

$dy = dx\frac{1}{\beta}$

thus $(*)$ becomes,

$\Gamma_{(\alpha)} = \int_0^\infty (\frac{x}{\beta})^{\alpha-1}e^{(-\frac{x}{\beta})}(\frac{1}{\beta})dx$

$1 = \int_0^\infty \frac{1}{\Gamma_{(\alpha)}\beta^\alpha}x^{\alpha-1}\ e^{(-\frac{x}{\beta})}dx.$

since $\alpha > 0$, $\beta > 0$ and $\Gamma_{(\alpha)} > 0$.

Gamma Distribution

Let x be the random variable probability density function of x is

$$F(x) = \frac{1}{\Gamma_{(\alpha)}\beta^\alpha}x^{\alpha-1}e^{-\frac{x}{\beta}}\ ,\ 0 < x < \infty.$$

$$F(x) = 0, \text{ else where.}$$

This P.D.F. of F(x) is called Gamma Distribution $[x \sim \alpha, \beta]$.

Moment Generating Function of Gamma Distribution:

$$M(t) = E[e^{tx}] = \int_0^\infty e^{tx} f(x)dx$$

$$= \int_0^\infty e^{tx} \frac{x^{\alpha-1} e^{\frac{-x}{\beta}}}{\Gamma_{(\alpha)}\beta^\alpha} dx$$

$$= \int_0^\infty \frac{x^{\alpha-1}}{\Gamma_{(\alpha)}\beta^\alpha} e^{x(t-\frac{1}{\beta})} dx$$

$$M(t) = \int_0^\infty \frac{x^{\alpha-1}}{\Gamma_{(\alpha)}\beta^\alpha} e^{-(\frac{(1-\beta t)x}{\beta})} dx$$

$$Let y = \frac{-(1-\beta t)}{\beta}\text{dx}$$

$$dy = \frac{-(1-\beta t)}{\beta}\ \text{dx}$$

$$M(t) = \int_0^\infty (\frac{\beta}{1-\beta t}) \frac{1}{\Gamma_{(\alpha)}\beta^\alpha} (\frac{\beta y}{1-\beta t})^{\alpha-1} e^{-y} dy$$

$$= (\frac{1}{1-\beta t})^\alpha \int_0^\infty \frac{1}{\Gamma_{(\alpha)}} \text{y}^{\alpha-1} dy e^{-y}$$

$$M(t) = (\frac{1}{1-\beta t})^\alpha (1),\ t < \frac{1}{\beta}$$

$$Now, M^{'(t)=(-\alpha)(1-\beta t)^{-\alpha-1}(-\beta)}$$

$$= (\alpha\beta)(1-\beta t)^{-\alpha-1}$$

$$M^{''}(t) = (\alpha\beta)(-\alpha-1)(1-\beta t)^{-\alpha-2}(-\beta)$$

$$= (\alpha\beta^2)(\alpha+1)(1-\beta t)^{-\alpha-2}$$

$$M^{'}(0) = \alpha\beta(1-\beta(0))^{-\alpha-1}$$

$$\therefore \text{Mean } \mu = \alpha\beta$$

$$M^{''}(0) = \alpha\beta^2(\alpha+1)(1-\beta(0))^{-\alpha-2}$$

$$= \alpha\beta^2(\alpha+1)$$

$$\text{Variance } \sigma^2 = M^{''}(0) - (M^{'}(0))^2$$

$$= \alpha\beta^2(\alpha+1) - \alpha^2\beta^2$$

$$= \alpha^2\beta^2 + \alpha\beta^2 - \alpha^2\beta^2$$

$$\therefore \sigma^2 = \alpha\beta^2$$

The mean of Gamma distribution is $\alpha\beta$.

The variance of Gamma distribution is $\alpha\beta^2$.

F(x) is a P.D.F:

Let $f(x) = \frac{1}{\Gamma_{(x)}\beta^\alpha} x^{\alpha-1} e^{\frac{-x}{\beta}}, 0 < x < \infty, f(x) \geqslant 0$

Now, $\int_0^\infty f(x)dx = \int_0^\infty \frac{1}{\Gamma_{(x)}\beta^\alpha} x^{\alpha-1} e^{\frac{-x}{\beta}} dx$

Let y=$\frac{x}{\beta} \Rightarrow \ dy = \frac{dx}{\beta}$

$$\int_0^\infty f(x)dx = \int_0^\infty \frac{\beta}{\Gamma_{(x)}\beta^\alpha} (\beta y)^{\alpha-1} e^{-y} dy$$

$$= \int_0^\infty \frac{1}{\Gamma_{(x)}} y^{\alpha-1} e^{-y} dy$$

$$= \frac{1}{\Gamma_{(\alpha)}} \Gamma_{(\alpha)}$$

$$\int_0^\infty f(x)dx = 1$$

$\therefore$ f(x) is a P.D.F.

Example:

If $(1-2t)^{-6}$, $t < \frac{1}{2}$ is the M.G.F. of the random variable X. Find μ and σ^2.

Solution:

The M.G.F. of the Gamma distribution is $M(t) = \frac{1}{(1-\beta t)^\alpha}, t < \frac{1}{\beta}$.

Given $M(t) = \frac{1}{(1-2t)^6}, t < \frac{1}{2}$

Then $\beta = 2, \alpha = 6, \mu = 12$ and $\sigma^2 = 24$.

Problem:

Let X be a random variable such that $E[X^m] = \frac{(m+3)!3^m}{3!}$ where $m = 1, 2, 3....$ Find the M.G.F. of X.

Solution:

Given $E[X^m] = \frac{(m+3)!3^m}{3!}$

To Find M.G.F:

$E[X^r]$=coefficient of $\frac{t^r}{r!}$ in M(t)

$M(t) = E[X^0]\frac{t^0}{0!} + E\left[X'\right]\frac{t'}{1!} + ... + E\left[X^r\right]\frac{t^r}{r!} + ...$

$= 1 + \frac{4!}{3!}3^1\frac{t}{1!} + \frac{5!}{3!}3^2\frac{t^2}{2!} + \frac{6!}{3!}3^3\frac{t^3}{2!} + ...$

$= 1 + \binom{4}{3}(3t)^1 + \binom{5}{3}(3t)^2 + \binom{6}{3}(3t)^3 + ... + \binom{r}{3}(3t)^r$

$= 1 + \binom{4}{1}(3t) + \binom{5}{2}(3t^2) + \binom{6}{3}(3t^2) + ...$

$M(t) = (1-3t)^{-4}$ which is the M.G.F. of Gamma distribution with $\alpha = 4, \beta = 3$.

$\therefore$ P.D.F. of X is f(x)=$\frac{1}{\Gamma_{(4)}3^4}x^3e^{\frac{-x}{3}}, 0 < x < \infty$.

Exponential Distribution

Let $\alpha = k$ and $\beta = \frac{1}{\alpha}$ and $x = w, \alpha = 1, \beta = \frac{1}{\lambda}$.

The P.D.F. of the exponential distribution is,

$g(w) = \frac{1}{\Gamma_{(1)}\beta}w^{1-1}e^{-\frac{w}{1/\lambda}}$

$= \frac{1}{1/\lambda} e^{-\lambda w} = \lambda e^{-\lambda w}, 0 < w < \infty.$

$g(w) = \lambda e^{-\lambda w}.$

To Prove: g(w) is a P.D.F.

$$g(w) = \lambda e^{-\lambda w}, \lambda \geqslant 0$$

$$\int_0^\infty g(w) dw = \int_0^\infty \lambda e^{-\lambda w} dw$$

$$= \lambda \left[\frac{e^{-\lambda w}}{-\lambda} \right]_0^\infty$$

$$= \lambda \left[\frac{-e^{-\infty} + e^0}{-\lambda} \right]_0^\infty$$

$$= \lambda \left[\frac{-e^{-\infty} + e^0}{\lambda} \right]$$

$$\int_0^\infty g(w) dw = 1$$

$$\therefore g(w) \text{ is a P.D.F}$$

M.G.F. of Exponential Distribution:

$M(t) = E\left[e^{tw}\right] = \int_0^\infty e^{tw} g(w) dw$

$$= \int_0^\infty e^{tw} e^{-\lambda w} \lambda dw$$

$$= \lambda \int_0^\infty e^{(t-\lambda) w} dw$$

$$= \lambda \left[\frac{e^{(t-\lambda)}}{t-\lambda} \right]_0^\infty$$

$$= \frac{\lambda}{t-\lambda}(-1) = \frac{-\lambda}{t-\lambda}$$

$$\therefore M(t) = \frac{-\lambda}{t-\lambda}$$

$$M(t) = -\lambda (t-\lambda)^{-1}$$

$$M'(t) = -\left[-\lambda (t-\lambda)^{-2} \right]$$

$$M'(t) = \lambda (t-\lambda)^{-2}$$

$$\Rightarrow M'(0) = \frac{\lambda}{(-\lambda^2} = \frac{1}{\lambda} = \mu$$

$$M''(t) = -2\lambda(t-\lambda)^{-3}$$

$$\Rightarrow M''(0) = \frac{-2\lambda}{(-\lambda)^{-3}} = \frac{2}{\lambda^2}$$

$$\sigma^2 = M''(0) - (M'(0))^2$$

$$= \frac{2}{\lambda^2} - (\frac{1}{\lambda})^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$$

$$= \frac{1}{\lambda^2}$$

$$\therefore \sigma^2 = \frac{1}{\lambda^2}$$

Problem:

Show that $\int_\mu^\infty \frac{1}{\Gamma_k} z^{k-1} e^{-z} dz = \sum_{x=0}^{k-1} \frac{\mu^x e^{-\mu}}{x!}, k = 1, 2, ...$

Solution:

$\int_\mu^\infty \frac{1}{\Gamma_k} z^{k-1} e^{-z} dz = \frac{1}{\Gamma_k} \int_\mu^\infty z^{k-1} d(e^{-z})$

$= \frac{1}{\Gamma_k} \left[(-e^{-z} z^{k-1})^\infty - \int_\mu^\infty -e^{-z}(k-1) z^{k-2} dz \right.$

$= \frac{1}{\Gamma_k} \left[e^{-\mu}\mu^{k-1} + \int_\mu^\infty e^{-z}(k-1) z^{k-2} dz \right]$

$= \frac{1}{\Gamma_k} \left[e^{-\mu}\mu^{k-1} + (k-1) \int_\mu^\infty z^{k-2} de^{-z} \right]$

$= \frac{1}{\Gamma_k} \left[e^{-\mu}\mu^{k-1} + (k-1)(-e^{-z} z^{k-2})_\mu^\infty - (k-1)(\int_\mu^\infty -e^{-z}(k-2) z^{k-3} dz) \right]$

$= \frac{1}{\Gamma_k} \left[e^{-\mu}\mu^{k-1} - (k-1) e^{-\mu}\mu^{k-2} + (k-1) \int_0^\infty -e^{-z}(k-2) z^{k-3} dz \right]$

$= \frac{\mu^{k-1} e^{-\mu}}{(k-1)!} + \frac{\mu^{k-2} e^{-\mu}}{(k-2)!} + \frac{\mu^{k-3} e^{-\mu}}{(k-3)!} + (k-1)(k-2)(k-3) \int_\mu^\infty e^{-z} z^{k-4} dz$

$= \frac{\mu^{k-1} e^{-\mu}}{(k-1)!} + \frac{e^{-\mu}\mu^{k-2}}{(k-2)!} + \frac{e^{-\mu}\mu^{k-3}}{(k-3)!} + ... + \frac{e^{-\mu}\mu^1}{1!} + \frac{e^{-\mu}\mu^0}{0!}$

$\therefore \int_\mu^\infty \frac{1}{\Gamma_k} z^{k-1} e^{-z} dz = \sum_{x=0}^{k-1} \frac{\mu^x e^{-\mu}}{x!}, k = 1, 2, 3, ...$

Chi-Square Distribution

In the P.D.F. of Gamma distribution take $\alpha = \frac{r}{2}, \beta = 2$.

Then $f(x) = \frac{1}{\Gamma_{(\frac{r}{2})} 2^{\frac{r}{2}}} x^{\frac{r}{2-1}} e^{\frac{-x}{2}}, 0 < x < \infty.$ f(x) is called the P.D.F. of χ^2 - distribution with the parameter and it is denoted by $\chi^2(r)$.

Note:

Here r is the degrees of freedom.

Problem:

If χ has the P.D.F. $F(x) = \frac{1}{4} x e^{\frac{-x}{2}}, 0 < x < \infty.$ Find the distribution of x also find the M.G.F. mean and variance.

Solution:

Given $f(x) = \frac{1}{4} x e^{\frac{-x}{2}}, 0 < x < \infty.$

$f(x) = \frac{1}{\Gamma_{(\frac{4}{2})} 2^{\frac{4}{2}}} x^{\frac{4}{2}-1} e^{-x/2}$ which is the P.D.F. of χ^2 - distribution.

Here $r = 4$, mean $r = 4$.

M.G.F. $M(t) = (1 - 2t)^{\frac{-4}{2}} = (1 - 2t)^{-2}$.

M.G.F. of χ^2 Distribution:

$M(t) = E\left[e^{tx}\right] = \int_0^\infty e^{tx} \frac{1}{\Gamma_{(\frac{r}{2})} 2^{r/2}} e^{-x/2} dx$

$= \frac{1}{\Gamma_{(r/2)} 2^{r/2}} \int_0^\infty e^{tx} x^{r-2/2} e^{-x/2} dx$

$= \frac{1}{\Gamma_{(r/2)} 2^{r/2}} \int_0^\infty e^{x(1/2-t)} x^{r-2/2} dx$

$= \frac{1}{\Gamma_{r/2} 2^{r/2}} \int_0^\infty e^{-x(1-2t/2)} x^{r-2/2} dx$

Let $y = \frac{x(1-2t)}{2} \Rightarrow dy = dx \frac{1-2t}{2}$

$M(t) = \frac{1}{\Gamma_{(r/2)} 2^{r/2}} \int_0^\infty e^{-y} (\frac{2y}{1-2t})^{r/2} dy (\frac{2}{1-2t})$

$= \frac{1}{\Gamma_{(r/2)} 2^{r/2}} \int_0^\infty (\frac{2}{1-2t})^{r/2} e^{-y} y^{r/2-1} dy$

$= \frac{1}{\Gamma_{(r/2)} 2^{r/2}} \Gamma_{(r/2)} (\frac{2}{1-2t})^{r/2}$

$= \frac{1}{2^{r/2}} (\frac{2}{1-2t})^{r/2}$

$= (\frac{1}{1-2t})^{r/2}$

$M^{'}(t) = \frac{-r}{2}(1-2t)^{-r/2-1}(-2)$

$M^{'}(t) = \frac{-r}{2}(1-2t)^{\frac{-r-2}{2}}(-2)$

$M^{'}(0) = \frac{-r}{2}(1)(-2)$

$M^{'}(0) = r$

$M^{''}(t) = \frac{-r}{2} \frac{-r-2}{2}(1-2t)^{\frac{-r-2}{2}-1}(-2)(-2)$

$M^{''}(t) = r \frac{-r-2}{2}(1-2t)^{\frac{-r-4}{2}}(-2)$

$M^{''}(t) = r(r+2)(1-2t)^{\frac{-r-4}{2}}$

$M^{''}(0) = r(r+2)$

$M^{''}(0) = r^2 + 2r$

variance $\sigma^2 = M^{''}(0) - (M^{'}(0))^2$

variance $= r^2 + 2r - r^2$

$\therefore$ variance $= 2r$

Problem:

If x is a $\chi^2(5)$ distribution determine the constants c and d so that $P_r(c < x < d) = 0.95$ and $P_r(x < c) = 0.025$.

Note: If the random variable x is $\chi^2(r)$ and if $c_1 < c_2$ then $P_r(c_1 \leqslant x \leqslant c_2) = P_r(x \leqslant c_2) - P_r(x_1)$.

Solution:

Given x is a random variable of $\chi^2(5)$.

Here $r = 5$.

$P_r(c < x < d) = 0.95$

$P_r(x < c) = 0.25$

From table, $P_r(x < 0.831) = 0.025$

$P_r(c < x < d) = P_r(x \leqslant d) - P_r(x < c)$

$0.95 = P_r(x \leqslant d) - 0.025$

$P_r(x < d) = 0.975$

$\therefore d = 12.8$

Problem:

Let x be $\chi^2(10)$ find the value of $P_r(3.25 \leqslant x \leqslant 20.5)$.

Solution:

$Given \mathrm{P}_r(3.25 \leqslant x \leqslant 20.5)$.

$\Rightarrow P_r(3.25 \leqslant x \leqslant 20.5) = P_r(x < 20.5) - P(x < 3.25)$

$= 0.975 - 0.025$

$P_r(3.25 \leqslant x \leqslant 20.5) = 0.95$

Problem:

Let X have a Gamma distribution with $\alpha = \frac{r}{2}$, where r is a positive integer $\beta > 0$. Define a random variable $Y = \frac{2X}{\beta}$. Find the P.D.F of Y.

Solution:

Given x have a Gamma distribution with $\alpha = \frac{r}{2}$. Now, P.D.F. of X is $F(x) = \frac{x^{\alpha-1}}{\Gamma_{(\alpha)}\beta^{\alpha}} e^{\frac{-x}{\beta}}$,

$\Rightarrow \frac{x^{\frac{r}{2}-1}}{\Gamma_{(\frac{r}{2})}\beta^{\frac{r}{2}}} x^{\frac{-x}{\beta}}, \beta > 0, 0 < x < \infty.$

Given $Y = \frac{2X}{\beta}$.

To Find: P.D.F. of Y.

The distribution function of Y is $G(Y) = Pr(Y \leqslant y) = P_r(\frac{2X}{\beta})$

$= P_r(X \leqslant \frac{\beta y}{2})$

$= \int_0^{\infty} \frac{1}{\Gamma_{(\frac{r}{2})}\beta^{\frac{r}{2}}} (\frac{\beta y}{2})^{\frac{r}{2}-1} e^{\frac{-y}{2}} \frac{\beta}{2} dy.$

The P.D.F. od Y is $g(Y) = G^{'}(Y)$

Thus P.D.F. of Y is given by $g(Y) = \frac{1}{\Gamma_{(\frac{r}{2})}\beta^{\frac{r}{2}}} \frac{\beta}{2} (\frac{\beta y}{2})^{\frac{r}{2}-1} e^{\frac{-y}{2}}$

$= \frac{1}{\Gamma_{(\frac{r}{2})}\beta^{\frac{r}{2}}} (\frac{\beta}{2})^{\frac{r}{2}} y^{\frac{r}{2}-1} e^{\frac{-y}{2}}$

$g(Y) = \frac{1}{\Gamma_{(\frac{r}{2})}2^{\frac{r}{2}}} y^{\frac{r}{2}-1} e^{\frac{-y}{2}}$, is the required P.D.F.

$\therefore Y \sim \chi^2(r)$.

Problem:

If X has Gamma distribution. Then $Y = \frac{2X}{\beta}$ has χ^2 - distribution with parameter, r.

Now, $Y = \frac{2X}{\beta} = \frac{2X}{4} = \frac{X}{2}$.

we know that $\alpha = \frac{r}{2}$

$P_r(3.28 < X < 25.2) = P_r(1.64 < \frac{X}{2} < 12.6)$

$= P_r(1.64 < Y < 12.6)$

$= P_r(Y < 12.6) - P_r(Y < 1.64)$

$= 0.95 - 0.5$

$P_r(3.28 < X < 25.2) = 0.90.$

Normal Distribution

Let X be the continuous random variable having the P.D.F,

$$f(x) = \begin{cases} \frac{1}{\sqrt{2\pi}b} e^{\frac{-1}{2}(\frac{x-a}{b})^2}, & -\infty < x < \infty, b > 0 \\ 0, & otherwise. \end{cases}$$

then X is said to be the normal distribution.

To Prove: f(x) is a P.D.F.

Let $I = \int_{-\infty}^{\infty} f(x)dx$

$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}b} e^{\frac{-1}{2}(\frac{x-a}{b})^2} dx$

$= \frac{1}{\sqrt{2\pi}b} \int_{-\infty}^{\infty} e^{\frac{-1}{2}}(\frac{x-a}{b})^2 dx$

put $y = \frac{x-a}{b} \Rightarrow dy = \frac{1}{b}dx$ as $-\infty < x < \infty, -\infty < y < \infty.$

$I = \frac{b}{\sqrt{2\pi}b} \int_{-\infty}^{\infty} e^{\frac{-1}{2}y^2} dy$

$I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-y^2}{2}} dy \qquad \longrightarrow (1)$

$consider$ I=$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-z^2}{2}} dz \qquad \longrightarrow (2)$

$(1) \times (2) \Rightarrow \mathrm{I}^2 = (\frac{1}{\sqrt{2\pi}})^2 [\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(\frac{y^2+z^2}{2})} dy\ dz]$

changing into polar co-ordinate by putting $y = r\ cos\theta, z = r\ sin\theta$

we get $|J|= r, dydz = rdrd\theta$.

$Also$, 0<r<∞ and $0 < \theta < 2\pi$.

$I^2 = \frac{1}{2\pi}\int_0^{2\pi}(\int_0^{\infty} e^{-t}dt)d\theta$

$= \frac{1}{2\pi}\int_0^{2\pi}[-e^{-t}]_0^{\infty}d\theta$

$= \frac{1}{2\pi}\int_0^{2\pi} 1d\theta$

$= \frac{1}{2\pi}[\theta]_0^{2\pi}$

$= \frac{1}{2\pi}2\pi$

$I^2 = 1$

Hence, f(x) is a P.D.F.

M.G.F. of Normal Distribution:

$M(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx}f(x)dx$

$= \int_{-\infty}^{\infty} e^{tx}\frac{1}{\sqrt{2\pi}b}e^{\frac{-1}{2}}(\frac{x-a}{b})^2dx$

$= \frac{1}{\sqrt{2\pi}b}\int_{-\infty}^{\infty} e^{tx}e^{\frac{-1}{2}}\frac{(x-a)^2}{b}dx$

$= \frac{1}{\sqrt{2\pi}b}\int_{-\infty}^{\infty} e^{\frac{-1}{2b^2}(x^2+a^2-2ax)e^{tx}}dx$

$= \frac{1}{\sqrt{2\pi}b}\int_{-\infty}^{\infty} e^{\frac{-1}{2b^2}(x^2+a^2-2ax+b^2t)}dx$

$[x-(a+b^2t)]^2 = x^2 + (a+b^2t)^2 - 2(a+b^2t)x$

$[x-(a+b^2t)]^2 = x^2 + a^2 + b^4t^2 + 2ab^2t - 2x(a+b^2t)$

$x^2 + a^2 - 2x(a+b^2t) = [x-(a+b^2t)^2 - b^4t^2 - 2ab^2t]$

$= [x-(a+b^2t)]^2 - b^2(b^2t^2 + 2at)$

$M(t) = \frac{1}{\sqrt{2\pi}b} \int_{-\infty}^{\infty} e^{\frac{-1}{2b^2}(x-(a+b^2t))^2} e^{(\frac{b^2t^2+2at}{2})} dx$

$M(t) = e^{\frac{b^2t^2+2at}{2}}$, [since, $\frac{1}{\sqrt{2\pi}b} \int_{-\infty}^{\infty} e^{\frac{-1}{2b^2}}[x-(a+b^2t)]^2 dx = 1$]

M(t)=e$^{\frac{at+b^2t^2}{2}} = e^{at}e^{\frac{b^2t^2}{2}}$

$\Rightarrow M^{'}(t) = e^{at}e^{\frac{b^2t^2}{2}}b^2t + e^{\frac{b^2t^2}{2}}ae^{at}$

$= e^{at}[tb^2e^{\frac{b^2t^2}{2}} + e^{\frac{b^2t^2}{2}}a]$

$M(0) = e^0(a)$

$M^{'}(0) = a$

$\therefore \mu = a$

$M^{''}(t) = e^{at}e^{\frac{b^2t^2}{2}}b^2 + (tb^2 + a)[e^{at}e^{\frac{b^2t^2}{2}}t^2b^2 + e^{\frac{b^2t^2}{2}}ae^{at}$

$M^{''}(0) = b^2 + a^2$

variance $\sigma^2 = M^{''}(0) - (M^{'}(0))^2$

$\Rightarrow \sigma^2 = b^2 + a^2 - a^2$

$\therefore \sigma^2 = b^2$

Note:

1. The P.D.F of normal distribution can be written as

   $f(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-1}{2}}(\frac{x-\mu}{\sigma})^2, -\infty < x < \infty.$

2. Normal distribution is denoted by $N(\mu, \sigma^2)$.

3. The M.G.F. of normal distribution is $e^{\mu t + \frac{\sigma^2t^2}{2}}$.

Standard Normal Distribution:

The normal distribution in which mean=0 and variance $\frac{1}{\mu}$ is called standard normal distribution.

The P.D.F. of standard normal distribution is given by,

$f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$.

The M.G.F. of standard normal distribution is

$M(t) = e^{\frac{t^2}{2}}$.

Properties of Normal Distribution:

1. The curve of the normal distribution is bell shaped one.

2. The normal curve is symmetrical about the vertical axis $x = \mu$.

3. The normal curve attains the maximum value at $x = \mu$.

4. In the normal distribution, Mean = Mode = Median.

5. For the normal distribution the measure of skewers is zero.

6. The curve has its points of inflection at $x = \mu \pm \sigma$ and it concave downwards if $\mu - \sigma < x < \mu + \sigma$ and concave upwards otherwise.

7. The total area under the curve above the horizontal axis is $\int_{-\infty}^{\infty} f(x)dx = 1$.

Theorem:

If the random variable X gas a $N(\mu, \sigma^2), \sigma^2 > 0$ then the random variable $W = \frac{X-\mu}{\sigma}$ is a $N(0,1)$.

Proof:

Given X is a $N(\mu, \sigma^2)$

The P.D.F. of X is given by, $f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-1}{2}(\frac{x-\mu}{\sigma})^2}, -\infty < x < \infty.$

Let $W = \frac{X-\mu}{\sigma}$

The distribution function of w is given by

$G(W) = P_r(W \leqslant w)$

$= P_r(\frac{x-\mu}{\sigma} \leqslant w)$

$= P_r(x \leqslant \mu + \sigma w)$

$= \int_{-\infty}^{\mu+\sigma w} f(x) dx$

$= \int_{-\infty}^{\mu+\sigma w} \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-1}{2}(\frac{x-\mu}{\sigma})^2} dx$

Let $W = \frac{X-\mu}{\sigma}$

$x = \mu + \sigma w \Rightarrow dx = \sigma dw$

$x = -\infty; w = -\infty$

$x = \mu + \sigma; w = \omega$

$G(w) = \int_{-\infty}^{\omega} \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-1}{2}w^2} \sigma dw$

$= \int_{-\infty}^{\omega} \frac{1}{\sqrt{2\pi}} e^{\frac{-w^2}{2}} dw$

The P.D.F. of w is $g(w) = G^{'}(W)$

$\therefore g(w) = \frac{1}{\sqrt{2\pi}} e^{\frac{-w^2}{2}}$, which is the P.D.F. of standard normal distribution.

$W \sim N(0,1)$.

Hence proved.

Note:

1. Let X be $N(0,1)$, we get the notation $N(X) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-w^2}{2}} dw$.

2. If X is a $N(\mu, \sigma^2)$ then $P_r(X < C_1) = N(\frac{C_1-\mu}{\sigma})$.

   Proof:

   Given $X - N(\mu, \sigma^2)$.

   Then $W = \frac{X-\mu}{}$ is a $N(0,1)$ (since, By above theorem)

   Now, $P_r(X < C_1) = P_r(\frac{X-\mu}{\sigma}, \frac{C_1-\mu}{\sigma})$

   $= P_r(W < \frac{C_1-\mu}{\sigma})$

   $= \int_{-\infty}^{\frac{C_1+\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{\frac{-w^2}{2}} dw$

   $\therefore P_r(X < C_1) = N(\frac{C_1-\mu}{\sigma})$

3. If X is a $N(\mu, \sigma^2)$, then $P_r(C_1 < X < C_2) = N(\frac{C_2-\mu}{\sigma}) - N(\frac{C_1-\mu}{\sigma})$

   If X is a $N(\mu, \sigma^2)$. Then $W = \frac{X-\mu}{\sigma}$ is a $N(0,1)$.

   $P_r(C_1 < X < C_2) = P_r(X < C_2) - P_r(X - C_1)$.

   $P_r(C_1 < X < C_2) = N(\frac{C_2-\mu}{\sigma}) - N(\frac{C_1-\mu}{\sigma})$.

Problem:

Let X be a $N(2, 25)$. Find i). $P_r(X < 10)$

ii). $P_r(0 < X < 10) = 0.6$

Solution:

Let X be a $N(2, 25)$.

$\mu = 2$, $\sigma^2 = 25 \Rightarrow \sigma = 5$.

we know that $X \sim N(\mu, \sigma^2)$.

Then $P_r(X < C) = N(\frac{C-\mu}{\sigma})$

i).$P_r(X < 10) = P_r(\frac{X-\mu}{\sigma} < \frac{10-\mu}{\sigma})$

$=\mathrm{P}_r(W < \frac{8}{5}) = P_r(W < 1.6)$

$= 0.945.$

ii) $P_r(0 < X < 10) = P_r(\frac{0-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{10-\mu}{\sigma})$

$= P_r(\frac{-2}{5} < W < \frac{8}{5}) = P_r(W < \frac{8}{5}) - P_r(W < \frac{-2}{5})$

$= P_r(W < 1.6) - P_r(W < -0.4)$

$= 0.945 - 1 + 0.655$

$= 0.6$

Problem:

Prove that $N(-x) = 1 - N(x)$.

Proof:

$N(-x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{\frac{-w^2}{2}} dw$

$= \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{\frac{-w^2}{2}} dw$

$= \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{\frac{-w^2}{2}} dw - \int_\infty^x \frac{1}{\sqrt{2\pi}} e^{\frac{-w^2}{2}} dw$

$= 1 - \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{\frac{-w^2}{2}} dw$

$\therefore N(-X) = 1 - N(X).$

Problem:

If X is a $N(\mu, \sigma^2)$ find b so that $Pr(-b < \frac{X-\mu}{\sigma} < b) = 0.90$

Solution:

Given P(-b$< \frac{X-\mu}{\sigma} < b) = 0.90$

P$(\mu - \sigma b < X < \mu + \sigma b) = 0.90$

We know that, $P(c_1 < X < c_2) = N(\frac{c_2-\mu}{\sigma}) - N(\frac{c_1-\mu}{\sigma})$ and

$N(-x) = 1 - N(x).$

$P(-b < \frac{X-\mu}{\sigma} < b) = N(b) - N(-b) = 0.90 = 2N(b) - 1 = 0.90$

$\Rightarrow 2N(b) = 1.90 \Rightarrow N(b) = \frac{1.9}{2} \Rightarrow N(b) = 0.95$

$\therefore b = 1.645$

Problem:

Let X be $N(\mu, \sigma^2)$. Then $P_r(\mu - 2\sigma < X < \mu + 2\sigma)$.

Solution:

$P_r(\mu - 2\sigma < X < \mu + 2\sigma) = P_r(\frac{\mu-2\sigma-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{\mu+2\sigma}{\sigma})$

$= P_r(-2 < W < 2) = N(2) - N(-2) = N(2) - 1 + N(-2)$

$= 2N(2) - 1 = 2(0.977) - 1 = 1.954 - 1 = 0.954$

$\therefore P_r(\mu - 2\sigma < X < \mu + 2\sigma) = 0.954.$

Problem:

Suppose that 10% of the probability for a certain distribution (i.e), $N(\mu, \sigma^2)$ is below 60 and that 5% is above 90 what are the values of μ and σ.

Solution:

Given X is a random variable, $X \sim N(\mu, \sigma^2)$ and

$P(X \leqslant 60) = 0.10$ and $P_r(X \leqslant 90) = 0.95$

$P_r(\frac{X-\mu}{\sigma} \leqslant \frac{60-\mu}{\sigma}) = 0.10$

$N(\frac{60-\mu}{\sigma}) = 0.10$

similarly, $N(\frac{90-\mu}{\sigma}) = 0.95$

$\frac{60-\mu}{\sigma} = -1.282$

$60 - \mu = -1.282\sigma$

$\mu - 1.282\sigma = 60 \longrightarrow (1)$

$\frac{90-\mu}{\sigma} = 1.645$

$90 - \mu = 1.645\sigma$

$\mu + 1.645\sigma = 90 \longrightarrow (2)$

$(2) - (1) \Rightarrow 2.927\sigma = 30$

$\sigma = 10.2$

substitute σ in (1) we get,$\mu - 1.282(10.2) = 60$

$\mu - 13.076 = 60$

$\mu = 60 + 13.076$

$\mu = 73.1$

$\therefore \mu = 73.1$ and $\sigma = 10.2$.

Problem:

If the random variable is $N(\mu, \sigma^2)$ where $\sigma^2 > 0$, then the random variable $Y = (\frac{X-\mu}{\sigma})^2$ is $\chi^2 - (1)$.

Solution:

Given X is a random variable, $X \sim N(\mu, \sigma^2)$.

Let $W = \frac{X\mu}{\sigma} \sim N(0, 1)$

$\therefore Y = W^2$

The distribution of Y is $G(Y) = P_r(Y \leqslant y)$

$= P_r(W^2 \leqslant y)$

$= P_r(W \leqslant \pm\sqrt{y})$

$G(Y) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{\frac{-w^2}{2}} dw$

$= 2 \int_0^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{\frac{-w^2}{2}} dw$

$t = w^2 \Rightarrow dt = 2wdw \Rightarrow dw = \frac{1}{2\sqrt{t}} dt$

$t = 0 \Rightarrow w = 0$

$t = \sqrt{y} \Rightarrow w = y$

$G(Y) = 2\int_0^y \frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}} \frac{1}{2\sqrt{t}} dt$

$= \int_0^y \frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}} t^{\frac{1}{2}-1} dt$

$= \int_0^y \frac{1}{\sqrt{2}\Gamma_{(\frac{1}{2})}} e^{\frac{-t^2}{2}} t^{\frac{1}{2}-1} dt$

$G(Y) = 1$

$\therefore G(Y) \sim X^2(1).$

Problem:

If e^{3t+8t^2} is the M.G.F. of random variable X find

$P_r(-1 < X < 9)$.

Solution:

Given e^{3t+8t^2} M.G.F. of random variable X.

$M(t) = e^{3t+8t^2}$

we know that $M(t) = e^{at+\frac{b^2t^2}{2}}$

Here $a = 3, b^2 = 16 \Rightarrow b = 4$.

$P_r(-1 < X < 9) = P_r(X < 9) - P_r(X < -1)$

$= N(\frac{9-3}{4}) - N(\frac{-1-3}{4})$

$= N(\frac{6}{4}) - N(\frac{-4}{4})$

$= N(1.5) - N(-1)$

$= 0.9332 - (1 - N(1))$

$= 0.9332 - 1 + .8413$

$\therefore P_r(-1 < X < 9) = 0.7745.$

Problem:

Show that the graph of the P.D.F. $N(\mu, \sigma^2)$ has the point of inflection $X = \mu \pm \sigma$.

Solution:

Given X is $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-1}{2}(\frac{x-\mu}{\sigma})^2} \longrightarrow (1), -\infty < x < \infty.$

The points of inflection of the normal curve are given by

$f''(x) \neq 0.$

Taking log on both sides in (1) we get,

$logf(x) = log(\frac{1}{\sqrt{2\pi}}) + loge^{\frac{-1}{2}}(\frac{x-\mu}{\sigma})^2$

$logf(x) = 0 - \frac{1}{2}(\frac{x-\mu}{\sigma})^2$

Differentiate with respect to 'x' we get,

$\frac{1}{f(x)} f'(x) = \frac{-1}{2} \frac{2(x-\mu)}{\sigma^2}$

$\frac{f'(x)}{f(x)} = \frac{-(x-\mu)}{\sigma^2}$

$f'(x) = \frac{-f(x)(x-\mu}{\sigma^2} \longrightarrow (2)$

$f'(x) = \frac{[e^{\frac{-1}{2}(\frac{x-\mu}{\sigma^2})^2}](x-\mu)}{\sqrt{2\pi}\sigma\sigma^2}$

$f'(x) = \frac{-e^{\frac{-1}{2}(\frac{x-\mu}{\sigma^2})^2}(x-\mu)}{\sqrt{2\pi}\sigma^3} \longrightarrow (3)$

Differentiate with respect to 'x' in (2) we get,

$f''(x) = \frac{-1}{\sigma^2}[f'(x)(x-\mu) + f(x)]$

$= \frac{-1}{\sigma^2}[\frac{-f(x)}{\sigma^2}(x-\mu)^2 + f(x)] \longrightarrow (4)$

$= \frac{-f(x)}{\sigma^2}[\frac{-(x-\mu)}{\sigma^2} + 1]$

$f^{''}(x) = \frac{1}{\sqrt{2\pi}\sigma^3} e^{\frac{-1}{2}(\frac{x-\mu}{\sigma^2})} [\frac{-(x-\mu)^2}{\sigma^2} + 1] \longrightarrow (5)$

Differentiate with respect to 'x' in (4) we get,

$f^{'''}(x) = \frac{-f(x)}{\sigma^4}(x-\mu)[\frac{-(x-\mu)^2}{\sigma^2} + 1] - \frac{2f(x)}{\sigma^4}(-1)(x-\mu)$

$= \frac{-f(x)}{\sigma^4}[\frac{-(x-\mu)^3}{\sigma^2} + (x-\mu) - 2(x-\mu)]$

$f^{'''}(x) = \frac{-f(x)}{\sigma^4}[\frac{-(x-\mu)}{\sigma^2} - (x-\mu)]$

$f^{'''}(x) = \frac{-1}{\sqrt{2\pi}\sigma} e^{\frac{-1}{2}(\frac{x-\mu}{\sigma})^2} [\frac{-(x-\mu)^3}{\sigma^2} - (x-\mu)] \longrightarrow (6)$

since $f^{''}(x) = 0$ we have,

$(4) \Rightarrow \frac{\frac{-1}{\sqrt{2\pi}\sigma} e^{\frac{-1}{2}(\frac{x-\mu}{\sigma})^2}}{\sigma^2} [\frac{-(x-\mu)^2}{\sigma^2} + 1] = 0$

$\Rightarrow [\frac{-(x-\mu)^2}{\sigma^2} + 1] = 0$

$\Rightarrow \frac{-(x-\mu)^2}{\sigma^2} = -1$

$\Rightarrow (x-\mu)^2 = \sigma^2$

$\Rightarrow (x-\mu) = \pm\sigma$

$x = \mu \pm \sigma$

substitute $x = \mu + \sigma$ in (6) we get,

$f^{'''}(x) = \frac{\frac{-1}{\sqrt{2\pi}\sigma^3}}{\sigma^4} e^{\frac{-1}{2}(\frac{\mu+\sigma-\mu}{\sigma})^2} [\frac{-(\mu+\sigma-\mu)^3}{\sigma^2} - (\mu+\sigma-\mu)]$

$= \frac{\frac{-1}{\sqrt{2\pi}\sigma^3}}{\sigma^4} e^{\frac{-1}{2\sigma}} [\frac{-\sigma^3}{\sigma^2} - \sigma]$

$= \frac{-1}{\sqrt{2\pi}\sigma^7} e^{\frac{-\sigma}{2}} [-2\sigma]$

$\neq 0$

The point of inflection is $x = \mu \pm \sigma$.

UNIT IV

Sampling Theory

Problem:

Let $X_1, X_2, ..., X_n$ denote the random sample of size n from a distribution. (i.e), $Y = X_1^2 + X_2^2$ find,

i). Distribution function of Y.

ii). P.D.F. of X.

Solution:

Given $X_1, X_2 \sim N(0,1)$.

The P.D.F. of X_1 is $f_1(x_1) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x62}{2}}, -\infty < x_1 < \infty$.

$\mathrm{X}_1 X_2$ are stochastically independent.

$\therefore$ the joint P.D.F. of X_1 and X_2 is

$f(X_1, X_2) = f(x_1)f(x_2) = \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}(x_1^2+x_2^2)}, -\infty < x_1 < \infty,$

$-\infty < x_2 < \infty.$

Let $Y = X_1^2 + X_2^2$ (sum of 2 random variables).

Let $G(y) = P_r(Y \leqslant y) = P_r(X_1^2 + X_2^2 \leqslant y)$

$= \iint_A f(x_1, x_2) dx_1 dx_2$, where $A = \{(x_1, x_2)/x_1^2 + x_2^2\}$

$= \iint_A \frac{1}{2\pi} e^{\frac{-1}{2}(x_1^2+x_2^2)} dx_1 dx_2$

changing into polar co-ordinate by $x_1 = rcos\theta$ and $x_2 = sin\theta$

$|J| = r$ and $dx_1 dx_2 = |J| dr d\theta = r dr d\theta$. $|J|$ is Jacobian.

Also, $x_1^2 + x_2^2 = r^2$ and $0 \leqslant \theta \leqslant 2\pi$.

$G(y) = \int_0^{\sqrt{y}} \int_0^{2\pi} \frac{1}{2\pi} e^{\frac{-r^2}{2}} r dr d\theta, t = r^2 \Rightarrow dt = 2rdr.$

$G(y) = \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_0^{\sqrt{y}} e^{\frac{-r^2}{2}} rdr$

put $r^2 = t \Rightarrow 2rdr = dt$

$r = 0 \Rightarrow t = 0$

$r = \sqrt{y} \Rightarrow t = y$

$G(Y) = \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_0^y e^{\frac{-1}{2}} \frac{dt}{2}$

$= \frac{1}{4\pi} [\theta]_0^{2\pi} \int_0^y e^{\frac{-1}{2}} dt$

$= \frac{2\pi}{4\pi} \int_0^y e^{\frac{-1}{2}} dt$

$= \frac{1}{2} \int_0^y e^{\frac{-1}{2}} dt$

$G(y) = \frac{1}{2} \int_0^y \frac{e^{\frac{-1}{2}} t^{\frac{2}{2}-1}}{\Gamma_{(\frac{2}{2})} 2^{\frac{2}{2}}} dt$

$G(y) = G'(y) = \frac{e^{\frac{-1}{2}} t^{\frac{2}{2}-1}}{\Gamma_{(\frac{2}{2})} 2^{\frac{2}{2}}} \sim \chi^2 - (n).$

Transformation of variables of discrete type:

Let X be a random variable of a discrete type having the P.D.F. f(x). Let $\mathcal{A}$ be the set of discrete points at each of discrete points in which $f(x) > 0$.

Let $y = u(x)$ defined a 1-1 transformation that maps $\mathcal{B}$ is the obtained by transforming each point in $\mathcal{A}$ in accordance with $y = u(x)$. We solve $y = u(x)$ for x in terms of y. Let $x = w(y)$ then for each y in $\mathcal{B}$ we have $x = w(t) \in \mathcal{A}$.

Consider the random variable $y = u(x)$ if y$\in \mathcal{B}$ then

$x = w(y) \in \mathcal{A}$.

Now the event Y=y. (i.e), $y = u(x)$ and $x = w(y)$ are equivalent.

Now, the P.D.F. of Y is

$g(y) = P_r(Y = y) = P_r(X = w(y)) = g(y) = f(w(y))$

Note:

Let $X_1, X_2, ..., X_n$ be the random sample of size n from a distribution that is $N(0,1)$ then $Y = X_1^2 + ... + X_n^2 \sim \chi^2(n)$.

Problem:

Let X have a binomial P.D.F. $f(x) = \begin{cases} \frac{3!}{x!(3-x)!}(\frac{2}{3})^x(\frac{1}{3})^{3-x} & x = 0,1,2,3 \\ 0 & elsewhere. \end{cases}$

find the P.D.F. of $Y = X^2$.

Solution:

Given $f(x) = \frac{3!}{x!(3-x)!}(\frac{2}{3})^x(\frac{1}{3})^{3-x}$ is the P.D.F. of X.

To find the P.D.F. of $Y = X^2$:

$\mathcal{A} = \{x/x = 0,1,2,3\}$

$\mathcal{B} = \{y/y = 0,1,4,9\}$

since there is no negative value in $\mathcal{A}, Y = X^2$ defines a $1-1$ transformation from $\mathcal{A}$ onto $\mathcal{B}$.

Now, $x = \sqrt{y}$ (i.e), $x = w(y) = \sqrt{y}$

P.D.F. of Y is $g(y) = f(w(y)) = f(\sqrt{y})$

$$g(y) = \begin{cases} \frac{3!}{x!(3-x)!}(\frac{2}{3})^x(\frac{1}{3})^{3-\sqrt{y}} & y = 0, 1, 4, 9 \\ 0 & elsewhere. \end{cases}$$

Transformation of two dimensional random variable of discrete type

Let X_1, X_2 be the two random variable of discrete type with joint P.D.F. $f(x_1, x_2)$.

Let $y_1 = u_1(x_1, x_2)$ and $y_2 = (x_1, x_2)$.

Define a 1-1 transformation between the points (x_1, x_2) of $\mathcal{A}$ and the points (y_1, y_2) of $\mathcal{B}$.

The equation $y_1 = u_1(x_1, x_2)$ and $y_2 = u_2(x_1, x_2)$ may be solved for x_1, x_2.

(i.e), $x_1 = w_1(y_1, y_2)$ and $x_2 = w_2(y_1, y_2)$.

Then the joint P.D.F. of Y_1 and Y_2 is given by

$g(y_1, y_2) = P_r(Y_1 = y_1, Y_2 = y_2)$

$= P(x_1 = w(y_1, y_2), x_2 = w(y_1, y_2))$

$= f(w_1(y_1, y_2), w_2(y_1, y_2))$

The marginal P.D.F. of Y_1 is $g_1(y_1) = \sum_{y_2} g(y_1, y_2)$ and the marginal P.D.F. of Y_2 is $g_2(y_2) = \sum_{y_1} g(y_1, y_2)$.

Problem:

Let X_1, X_2 be two independent random variable have the poisson distribution with parameter μ_1 and μ_2 respectively. Find the distribution of the random variable $Y = X_1 + X_2$. (or) If X_1, X_2 are independent poisson variable with mean $\mu_1 \& \mu_2$ respectively. Then $X_1 + X_2$ is also a poission variable with mean $\mu_1 + \mu_2$.

Solution:

Let X_1, X_2 be poission variable.

The P.D.F. of X_1 is $f_1(X_1) = \frac{\mu_1^{x_1} e^{-\mu_1}}{x_1!}, 0 < x_1 < \infty$

The P.D.F. of X_2 is $f_2(x_2) = \frac{\mu_2^{x_2} e^{-\mu_2}}{x_2!}, 0 < x_2 < \infty$.

The joint P.D.F. of X_1 and X_2 is, $f(x_1, , x_2) = f_1(x_1) f_2(x_2)$

$= \frac{\mu_1^{x_1} e^{-\mu}}{x_1!} \frac{\mu_2^{x_2} e^{-\mu_2}}{x_2!}$

Let $\mathcal{A} = \{(x_1, x_2)/x_1 = 0, 1, 2, ..., x_2 = 0, 1, 2.....\}$

and $\mathcal{B} = \{(y_1, y_2)/y_1 = 0, 1, 2, ..., y_2 = 0, 1, 2....\}$.

Let $Y_1 = X_1 + X_2$.

choose$Y_2 = X_2$ such that the transformation is 1-1.

$x_2 = y_2, x_1 = y_1 - y_2$

$w_1(y_1, y_2) = y_1 - y_2$, $w_2(y_1, y_2) = y_2$

The joint P.D.F. of Y_1 & Y_2 is,

$g(y_1, y_2) = f(w_1(y_1, y_2), w_2(y_1, y_2)) = f(y_1 - y_2, y_2), 0 < y_1, y_2 < \infty$.

The marginal P.D.F. of Y_1 is $g_1(y_1) = \sum_{y_2=0} g(y_1, y_2)$

$= \sum_{y_2=0} \frac{\mu_1^{y_1-y_2} e^{-\mu_1}}{(y_1-y_2)!} \frac{\mu_2^{y_2} e^{-\mu_2}}{y_2!}$

$= \mu_1^{y_1} e^{-(\mu_1+\mu_2)} \sum_{y_2=0}^{\infty} \frac{\mu_1^{-y_2} \mu_2^{y_2}}{(y_1-y_2)! y_2!}$.

Student t-distribution

Let W denote the random variable N(0, 1). Let V denote the random variable having $\chi^2 - (r)$. Suppose w and v are stochastically independent then the joint P.D.F. of w and v is given by

$$h(u, v) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{\frac{-w^2}{2}} \frac{1}{\Gamma_{(\frac{r}{2})} 2^{r/2}} v^{\frac{r}{2}-1} e^{\frac{-v}{2}}, & -\infty < w < \infty. \\ 0 & elsewhere. \end{cases}$$

since $f(w) = \frac{1}{\sqrt{2\pi}} e^{\frac{-w^2}{2}}$ and $g(v) = \frac{1}{\Gamma_{(\frac{r}{2})} 2^{\frac{r}{2}}} e^{\frac{-v}{2}} v^{\frac{r}{2}-1}$

Let $U = V$.

Now the function $t = \frac{w}{\sqrt{\frac{v}{r}}}$ and $u = v$ maps.

$\mathcal{A} = \{(w, v) / -\infty < w < \infty, 0 < v < \infty\}$ onto

$\mathcal{B} = \{(t, u) / -\infty < t < \infty, 0 < u < \infty\}$ and is 1-1 correspondence.

Let $w = t\sqrt{u/r}$ and $v = u$.

$$J = \begin{vmatrix} \frac{\partial w}{\partial t} & \frac{\partial w}{\partial u} \\ \frac{\partial v}{\partial t} & \frac{\partial v}{\partial u} \end{vmatrix} = \begin{vmatrix} \sqrt{\frac{u}{r}} & \frac{t}{2\sqrt{\frac{u}{r}}} \\ 0 & 1 \end{vmatrix} \Rightarrow |J| = \sqrt{\frac{u}{r}}$$

The joint P.D.F. of t and u is

$g(t,u) = h[w_1(y_1,y_2), w_2(y_1,y_2)] \cdot |J|$

$= h(t\sqrt{\frac{u}{r}}, u)(\sqrt{\frac{u}{r}})$

$= \frac{1}{\sqrt{2\pi}} e^{\frac{-t^2 u}{2r}} \frac{1}{\Gamma_{(\frac{r}{2})} 2^{r^2}} u^{\frac{r}{2}-1} e^{\frac{-u}{2}} \sqrt{\frac{u}{r}}$

$= \frac{1}{\sqrt{2\pi r}\Gamma_{(\frac{r}{2})} 2^{\frac{r}{2}}} u^{\frac{r+1}{2}-1} e^{\frac{-u}{2}(\frac{t^2}{r}+1)}, \infty < t < \infty, 0 < u < \infty.$

$$\text{Thus, } g(t,u) = \begin{cases} \frac{1}{\sqrt{2}\Gamma_{(\frac{r}{2})} 2^{\frac{r}{2}}} e^{\frac{-u}{2}(\frac{t^2}{r}+1)} u^{\frac{r+1}{2}-1}, & -\infty < t < \infty, 0 < u < \infty. \\ 0, & elsewhere. \end{cases}$$

The marginal P.D.F. of t is, $g_1(t) = \int_0^\infty g(t,u)du$

$= \int_0^\infty \frac{1}{\sqrt{2\pi r}\Gamma_{(\frac{r}{2})} 2^{\frac{r}{2}}} u^{\frac{r+1}{2}-1} e^{\frac{-u^2}{2}(\frac{t^2}{r}+1)} du$

$= \frac{1}{\sqrt{2\pi r}\Gamma_{(\frac{r}{2})} 2^{\frac{r}{2}}} \int_0^\infty u^{\frac{r+1}{2}-1} e^{\frac{-u}{2}(\frac{t^2}{r}+1)} du$

Let $y = \frac{u}{2}(\frac{t^2}{r}+1) \Rightarrow dy = \frac{1}{2}(1+\frac{t^2}{r})du$

$$\begin{aligned} g_1(t) &= \frac{1}{\sqrt{2\pi r}\Gamma_{(\frac{r}{2})} 2^{\frac{r}{2}}} \int_0^\infty (\frac{2y}{(1+\frac{t^2}{r})})^{\frac{r+1}{2}-1} e^{-y(\frac{2}{1+\frac{t^2}{r}})} dy \\ &= \frac{1}{\sqrt{2\pi r}\Gamma_{(\frac{r}{2})} 2^{\frac{r}{2}}} (\frac{2}{1+\frac{t^2}{r}})^{\frac{r+1}{2}} \int_0^\infty y^{\frac{r+1}{2}-1} e^{-y} dy \\ &= \frac{1}{\sqrt{2\pi r}\Gamma_{(\frac{r}{2})} 2^{\frac{r}{2}}} \frac{2^{\frac{r}{2}} 2^{\frac{1}{2}}}{(1+\frac{t^2}{2})^{\frac{r+1}{2}}} \Gamma_{(\frac{r+1}{2})} \\ g_1(t) &= \frac{\Gamma_{(\frac{r+1}{2})}}{\sqrt{\pi r}\Gamma_{(\frac{r}{2})}(1+\frac{t^2}{r})^{\frac{r+1}{2}}} \end{aligned}$$

The F-distribution

Let U and V be the random variable which belongs to χ^2- distribution with degrees of freedom r_2 and r_2 respectively.

(i.e), $U \sim \chi^2 - (r_1)$ and $V \sim^2 -(r_2)$.

Let U and V are stochastically independent variable.

Thus, the joint P.D.F. of U and V is

$$h(u,v) = \begin{cases} \frac{1}{\Gamma_{(\frac{r_1}{2})}\Gamma_{(\frac{r_2}{2})}2^{\frac{r_1+r_2}{2}}} u^{\frac{r_1}{2}-1} v^{\frac{r_2}{2}-1} e^{-(\frac{u+v}{2})}, & 0 < u < \infty, 0 < v < \infty. \\ 0, & elsewhere. \end{cases}$$

Now we define the new random variable $f = \frac{(\frac{U}{r_1})}{(\frac{V}{r_2})}$

The distribution having the random variable is called

"F - distribution" with parameters r_1 and r_2.

To find the P.D.F. of F-distribution:

Let $w = v$.

The function $f = \frac{r_2}{r_1}\frac{u}{v}$ and $w = v$ are 1-1 mapping from

$\mathcal{A} = \{(u,v)/0 < u < \infty, 0 < w < \infty\}$ and

$\mathcal{B} = \{(f,w)/0 < f < \infty, 0 < w < \infty\}$.

Now,u=$\frac{r_1}{r_2}fw$ and $v = w$.

$$J = \begin{vmatrix} \frac{\partial u}{\partial f} & \frac{\partial u}{\partial w} \\ \frac{\partial v}{\partial f} & \frac{\partial v}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{r_1}{r_2}w & \frac{r_1}{r_2}f \\ 0 & 1 \end{vmatrix} = \frac{r_1}{r_2}w$$

$|J| = \frac{r_1}{r_2}w$

The joint P.D.F. of f and w is $g(f,w) = h(\frac{r_1}{r_2}fw, w)|J|$

$$g(f,w) = \frac{1}{\Gamma_{(\frac{r_1}{2})}\Gamma_{(\frac{r_2}{2})}2^{\frac{r_1+r_2}{2}}}(\frac{r_1}{r_2}fw)^{\frac{r_1}{2}-1} e^{\frac{-w}{2}(\frac{r_1}{r_2}f+1)}(\frac{r_1}{r_2}w)w^{\frac{r_2}{2}-1}$$

The marginal P.D.F. of f is,

$$g_1(f) = \frac{(\frac{r_1}{r_2})^{\frac{r_1}{2}} f^{\frac{r_1}{2}-1}}{\Gamma_{(\frac{r_1}{2})}\Gamma_{(\frac{r_2}{2})} 2^{\frac{r_1+r_2}{2}}} \int_0^\infty w^{\frac{r_1+r_2}{2}-1} e^{\frac{-w}{2}(\frac{r_1}{r_2}f+1)} dw$$

put $y = \frac{w}{2}(\frac{r_1}{r_2}f+1) \Rightarrow dy = \frac{dw}{2}(\frac{r_1}{r_2}f+1)$

$$g_1(f) = \frac{(\frac{r_1}{r_2})^{\frac{r_1}{2}} f^{\frac{r_1}{2}-1}}{\Gamma_{(\frac{r_1}{2})}\Gamma_{(\frac{r_2}{2})} 2^{\frac{r_1+r_2}{2}}} \int_0^\infty (\frac{2y}{\frac{r_1}{r_2}f+1})^{\frac{r_1+r_2}{2}-1} e^{-y} \frac{2dy}{\frac{r_1}{r_2}f+1}$$

$$= \frac{(\frac{r_1}{r_2})^{\frac{r_1}{2}} f^{\frac{r_1}{2}-1}}{\Gamma_{(\frac{r_1}{2})}\Gamma_{(\frac{r_2}{2})} 2^{\frac{r_1+r_2}{2}}} \int_0^\infty y^{\frac{r_1+r_2}{2}-1} e^{-y} dy$$

$$= \frac{(\frac{r_1}{r_2})^{\frac{r_1}{2}} f^{\frac{r_1}{2}-1}}{\Gamma_{(\frac{r_1}{2})}\Gamma_{(\frac{r_2}{2})} 2^{\frac{r_1+r_2}{2}}} (\frac{2}{\frac{r_1}{r_2}f+1})^{\frac{r_1+r_2}{2}} \Gamma_{(\frac{r_1+r_2}{2})}$$

$$g_1(f) = \frac{(\frac{r_1}{r_2})^{\frac{r_1}{2}} f^{\frac{r_1}{2}-1} \Gamma_{(\frac{r_1+r_2}{2})}}{\Gamma_{(\frac{r_1}{2})}\Gamma_{(\frac{r_2}{2})} (\frac{r_1}{r_2}f+1)^{\frac{r_1+r_2}{2}}}, 0 < f < \infty.$$

Definition:

A function of 1 (or) more random variables that does not depend upon any unknown parameter is called a 'statistic'.

In accordance with this definition the random variable $Y = \sum_{i=1}^{n} X_i$ discuss about the statistic. But the random variable $Y = \frac{(X_1-\mu)}{\sigma}$ is not a statistic, unless μ and σ are known numbers. It should be noted that, although a statistic does not depend upon any unknown parameter, the distribution of that statistic may very well depend upon unknown parameters.

Let the random variable X_i be the function of the i^{th} outcome $i = 1, 2, ..., n$. Then $X_1, X_2, ..., X_n$ the items of a random sample from the distribution under consideration. Suppose that we can define a statistic $Y = u(X_1, X_2, ..., X_n)$ whose P.D.F. is found to by $g(y)$.

This P.D.F. shows that there is a great probability that Y has a value close to the unknown parameter.

Definition:

Let $X_1, X_2, ..., X_n$ denote n mutually stochastically independent random variables, each of which has the same but possibly unknown P.D.F. f(x) that is, the probability density functions of $X_1, X_2, ..., X_n$ respectively, $f_1(x) = f(x_1), f_2(x) = f(x_2), ...$ $......., f_n(x) = f(x_n)$.

So that the joint P.D.F. $f(x_1)f(x_2)......f(x_n)$. The random variable $X_1, X_2, ..., X_n$ are them said tho constitute, a " random sample" from a distribution that has a P.D.F. f(x).

Definition:

Let $X_1, X_2, ..., X_n$ denote random sample size n from a given distribution. The statistic $\bar{X} = \frac{X_1+X_2+....+X_n}{n} = \sum_{i=1}^{n} \frac{X_i}{n}$ is called the mean of the random sample and the statistic $s^2 = \sum_{i=1}^{n} \frac{(X_i-\bar{X})^2}{n} = \sum_{i=1}^{n} \frac{X_i^2}{n} - (\bar{X})^2$ is called the variable of the random sample.

The distribution of the sample defined to be the distribution obtained by assigning a probability of $\frac{1}{n}$ to each of the points $x_1, x_2, ..., x_n$. This is the distribution of discrete type.

The corresponding distribution function will be denoted by $F_n(x)$ and it is a step function.

The function $F_n(x)$ is often called the "empirical distribution function".

Random sampling distribution theory means the general problem of finding distributions of functions of the in terms of a random sample. Up to this point, the only method, other than direct probabilistic arguments, of finding the distribution of a function of one or more random variables is the "distribution function technique". (i.e), If $X_1, X_2, ..., X_n$ are random variables, the distribution of $Y = u(X_1, X_2, ..., X_n)$ is determined by computing the distribution function of Y, $G(y) = P_r[u(X_1, X_2, ..., X_n) \leqslant y]$.

Problem:

Let X_1 and X_2 be stochastically independent random variable $Y_1 = u_1(x_1)$ and $Y_2 = u_2(x_2)$. Show that Y_1 and Y_2 are stochastically independent.

Solution:

Let X_1 and X_2 be two random variables of continuous type with joint P.D.F. $f(x_1, x_2)$.

since X_1 and X_2 are stochastically independent.

$\therefore f(x_1, x_2) = f(x_1)f(x_2)$

Let $y_1 = u(x_1)$ and $y_2 = u(x_2)$ be a function of X_1 alone and X_2 alone.

This is a 1-1 transformation from $\mathcal{A}$ to $\mathcal{B}$ where $\mathcal{A}$ is a square of (x_1, x_2) and $\mathcal{B}$ is a space of (y_1, y_2).

The inverse function of Y_1 and Y_2 are $X_1 = w_1(y_1)$ and $X_2 = w_2(y_2)$

$$J = \begin{vmatrix} w_1^{'}(y_1) & 0 \\ 0 & w_2(y_2) \end{vmatrix} = w_1^{'}(y_1)w_2^{'}(y_2) \neq 0.$$

The joint P.D.F. of Y_1 and Y_2 are

$$g(y_1, y_2) = \begin{cases} f_1(w_1, y_1), f_(w_2, y_2), & |J| \neq 0 \\ \\ 0, & elsewhere. \end{cases}$$

The marginal P.D.F. of Y_1 and Y_2 are

$g(Y_1) = f_1[w_1(y_1)]w_1^{'}(y_1)$

$g_2(Y_2) = f_2[w_2(y_2)]w_2^{'}(y_2)$

$g(y_1, y_2) = g_1(y_1)g_2(y_2)$

$\therefore Y_1$ and Y_2 are stochastically independent.

Problem:

Let X_1 and X_2 be random sample from the normal distribution N(0, 1). Show that the marginal P.D.F. of $Y_1 = \frac{X_1}{X_2}$ is a cauchy P.D.F. $g_1(y_1) = \frac{1}{\pi(1+y)^2}, -\infty < y_1 < \infty$.

Solution:

The joint P.D.F. of X_1 and X_2 are $f(x_1, x_2) = f_1(x_1)f_2(x_2)$

$= \frac{1}{\sqrt{2\pi}}e^{\frac{-x_1^2}{2}}\frac{1}{\sqrt{2\pi}}e^{\frac{-x_2^2}{2}}$

Hence the two dimensional space $\mathcal{A}$ is the $x_1 - x_2$ plane.

$\mathcal{A} = \{(x_1, x_2)/ -\infty < x_1, x_2 < \infty\}$

Let $X_2 = Y_2$.

Then $Y_1 = \frac{X_1}{X_2}$ and $Y_2 = X_2$

This represent a 1-1 transformation that map $\mathcal{A}$ onto $\mathcal{B}$.

$\mathcal{B} = \{(y_1, y_2)/-\infty < y_1, y_2 < \infty\}$

$Y_1 = u_1(x_1, x_2) = \frac{X_1}{X_2}$

$y_2 = U_2(x_1, x_2) = X_2$

Then $x_1 = y_1 y_2$ and $x_2 = y_2$

$$|J| = \begin{vmatrix} y_2 & y_1 \\ 0 & 1 \end{vmatrix} = y_2 \neq 0.$$

Joint P.D.F. of Y_1 and Y_2 is

$g(y_1, y_2) = f[w_1(y_1, y_2), w_2(y_1, y_2)]|J|$

$= f(y_1 y_2, y_2) y_2 = \frac{1}{2\pi} e^{\frac{-1}{2}(y_1^2 y_2^2 + y_2^2)} y_2, -\infty < y_1 y_2 < \infty.$

The marginal P.D.F. of Y_1 is,

$g_1(y_1) = \int_0^\infty g(y_1, y_2) y_2 dy_2.$

$= \frac{2}{2\pi} \int_0^\infty e^{\frac{-1}{2}} y_2^2 (1 + y_1^2) y_2 dy_2.$

put $t = \frac{y_2^2}{2}(1 + y_1^2)$

$dt = \frac{1}{2} 2 y_2 (1 + y_1^2) dy_2$

$g_1(y_1) = \frac{1}{\pi} \int_0^\infty e^{-t} \frac{dt}{(1+y_1^2)^2}$

$= \frac{1}{\pi(1+y_1^2)} \int_0^\infty e^{-t} dt$

$= \frac{1}{\pi(1+y_1^2)} [e^{-t}]_0^\infty$

$= \frac{1}{\pi(1+y_1^2)} (0 + 1), -\infty < y_1 < \infty.$

$= \frac{1}{\pi(1+y_1^2)}.$

which is a P.D.F. of cauchy distribution.

Problem:

Determine c for each f(x) is a beta P.D.F. of $f(x) = cx(1-x^3)$, $0 < x < 1$.

Solution:

$$\text{Given } f(x) = \begin{cases} cx(1-x)^3, & 0 < x < 1 \\ \\ 0, & elssewhere. \end{cases}$$

since f(x) is P.D.F. $\int_0^1 cx(1-x)^3 dx = 1$.

put $y = 1-x, \Rightarrow dy = -dx$.

$\Rightarrow \int_1^0 -c(1-y)y^3 dy = 1$

$\Rightarrow -c\int_1^0 (y^3 - y^4)dy = 1$

$\Rightarrow -c[\frac{y^4}{4} - \frac{y^5}{5}]_1^0 = 1$

$\Rightarrow -c[0 - (\frac{1}{4} - \frac{1}{5})] = 1$

$\Rightarrow -c[\frac{-1}{4} + \frac{1}{5}] = 1$

$\Rightarrow -c[\frac{-5+4}{20}] = 1$

$\Rightarrow \frac{c}{20} = 1$

$\Rightarrow c = 20.$

Unit -V

Distribution of and $\frac{ns^2}{\sigma^2}$

Let $X_1, X_2, ..., X_n$ denote random sample of size n from distribution.

(i.e), $n(\mu, \sigma^2) = \frac{\sum_{i=1}^{n} X_i}{n}$ is a mean of random sample.

$S^2 = \sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{n}$ is variance of random sample.

Then (i).$\bar{X}$ is $n(\mu, \sigma^2)$,

(ii). $\frac{ns^2}{\sigma^2}$ is $\chi^2 - (n, 1)$.

(iii). $\bar{X}$ and S^2 are stochastically independent.

Proof:

(i). $\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$

(i.e), $\bar{X} = \sum_{i=1}^{n} K_i X_i$ where $K_i = \frac{1}{n}$ $\forall i$.

since, $X_1, X_2, ..., X_n$ is a random sample from a distinct that is $n(\mu, \sigma^2)$.

putting $\mu_1 = \mu_2 = ... = \mu_n = \mu$, $\sigma_1^2 = \sigma_2^2 = ... = \sigma_n^2 = \sigma$,

$K_1 = K_2 = ... = K_n = \frac{1}{n}$.

By theorem (1),

$\bar{X}$ is normal with mean $\sum_{i=1}^{n} K_i \mu_i = \mu$ and variance $\sum_{i=1}^{n} K_i \sigma_i^2 = \frac{\sigma^2}{n}$.

(i.e), $\bar{X}$ is $n(\mu, \frac{\sigma^2}{n})$.

The P.D.F. of $\bar{X}$ is $\frac{\sqrt{n}}{\sqrt{2\pi}\sigma} e^{-\frac{n(\bar{X}-\mu)^2}{2\sigma^2}}$.

(ii). To prove, $\frac{nS^2}{\sigma^2}$ is $\chi^2 - (n-1)$.

Let $Y_1 = \bar{X}$.

$Y_1 = \frac{x_1+x_2+......+x_n}{n}, Y_2 = X_2, Y_3 = X_3, ..., Y_n = X_n$.

so that corresponding transformation define a 1-1 transformation.

The inverse function of the transformation are

$x_1 = ny_1 - x_2 - x_3 - .... - x_n = ny_1 - y_2 - y_3 - ..... - y_n$.

$x_2 = y_2; x_3 = y_3; ....; x_n = y_n$.

The Jacobian of the transformation is

$$J = \begin{vmatrix} n & -1 & -1 & .... & -1 \\ 0 & 1 & 0 & .... & 0 \\ 0 & 0 & 1 & .... & 0 \\ ..... & .... & ..... & .... & ...... \\ 0 & 0 & 0 & .... & 1 \end{vmatrix}$$

(i.e), $J = n$

$\Rightarrow |J| = n$

The joint P.D.F. of $Y_1, Y_2, ..., Y_n$ is

$g(y_1, y_2, ..., y_n) = |J| f(ny_1 - y_2 - ..... - y_n, y_2, y_3, ..., y_n)$

where, $f(x_1, x_2, ..., x_n) = (\frac{1}{\sqrt{2\pi}})^n e^{-\sum_{i=1}^{n} \frac{(X_i-\mu)^2}{2\sigma^2}}$

consider, $\sum_{i=1}^{n}(X_i - \mu)^2 = \sum_{i=1}^{n}[(X_i - \bar{X}) + (\bar{X} - \mu)]^2$

$\sum_{i=1}^{n}(x_i - \mu)^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2 +$

$n(\bar{x}-\mu)^2 + [\sum_{i=1}^{n}(x_i-\bar{x})2(\bar{x}-\mu)] \longrightarrow (1).$

But $\sum_{i=1}^{n}(x_i-\bar{x}) = \sum_{i=1}^{n} x_i - n\bar{x} = n\bar{x} - n\bar{x}.$

$\sum_{i=1}^{n} = (x_i-\bar{x}) = 0.$

$(1) \Rightarrow \sum_{i=1}^{n}(x_i-\mu)^2 = \sum_{i=1}^{n}(x_i-\bar{x})^2 + n(\bar{x}-\mu)^2.$

$f(x_1, x_2, ..., x_n) = (\frac{1}{\sigma\sqrt{2\pi}})^n e^{-\frac{[\sum_{i=1}^{n}(x_i-\bar{x})^2 + n(\bar{x}-\mu)^2]}{2\sigma^2}}$ and

$g(y_1, y_2, ..., y_n) = n(\frac{1}{\sqrt{2\pi}\sigma})^n e^{-\frac{(ny_1-y_2-....-y_n-y_n)^2}{2\sigma^2}} \sum_{i=2}^{n} \frac{(y_i-y_1)^2}{2\sigma} e^{-\frac{n(y_1-\mu)^2}{2\sigma^2}}$

The conditional P.D.F. of $Y_2, Y_3, ..., Y_n$ given $Y_1 = y_1$ is,

$\phi(y_2, y_3, ..., y_n/y_1) = \sqrt{n}(\frac{1}{\sigma\sqrt{2\pi}})^{n-1} e^{-\frac{q}{2\sigma^2}}.$

where $q = ((n-1)y_1 - y_2 - y_3.... - y_n)^2 + \sum_{i=2}^{n}(y_i - y_1)^2.$

since the P.D.F. of $Y_1 = \bar{X}$ is $\frac{\sqrt{n}}{\sigma\sqrt{2\pi}} e^{\frac{-n(y_i-y)^2}{2\sigma^2}}.$

since $\phi(y_2, y_3, ..., y_n/y_n)$ is the conditional joint P.D.F,

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ...... \int_{-\infty}^{\infty} \phi(y_2, y_3, ..., y_n/y_1) dy_2 dy_3 .....dy_n = 1 \longrightarrow (2)$

consider $S^2 = \frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{n} \Rightarrow ns^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2$

$= (X_1 - \bar{X})^2 + \sum_{i=2}^{n}(X_i - \bar{X})^2$

$= ((n-1)y_1 - y_2 - ...... - y_n)^2 + \sum_{i=2}^{n}(Y_i - Y_1)^2$

$ns^2 = Q \Rightarrow \frac{ns^2}{\sigma^2} = \frac{Q}{\sigma^2}$

The conditional M.G.F. of $\frac{ns^2}{\sigma^2}$ is given by $Y_1 = y_1$ is

$m(t) = E[\frac{e^{\frac{t\theta}{\sigma^2}}}{y_1}]$

$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ...... \int_{-\infty}^{\infty} \sqrt{n}(\frac{1}{2\pi\sigma^2})^{n-1} e^{-\frac{q}{2\sigma^2}} e^{\frac{tq}{\sigma^2}} dy_2 dy_3 ......dy_n.$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ...... \int_{-\infty}^{\infty} \sqrt{n}(\frac{1}{2\pi\sigma^2})^{\frac{n-1}{2}} e^{-\frac{(1-2t)q}{2\sigma^2}} dy_2 dy_3 ......dy_n.$

$= (\frac{1}{1-2t})^{\frac{n-1}{2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ...... \int_{-\infty}^{\infty} \sqrt{n}(\frac{1-2t}{2\pi\sigma^2})^{\frac{n-1}{2}} e^{-\frac{(1-2t)q}{2\sigma^2}} dy_2 dy_3 ....dy_n \rightarrow (3).$

In (3) replacing σ^2 by $\frac{\sigma^2}{1-2t}$. we get the integral,

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}.....\int_{-\infty}^{\infty}\sqrt{n}(\frac{1-2t}{2\pi\sigma^2})^{\frac{n-1}{2}}e^{-\frac{(1-2t)q}{2\sigma^2}}dy_2dy_3......dy_n = 1.$

$\therefore M(t) = (\frac{1}{1-2t})^{\frac{n-1}{2}} = (1-2t)^{-(\frac{n-1}{2})}, t > \frac{1}{2}.$

Thus, the conditional M.G.F. of $\frac{nS^2}{\sigma^2}$ given

$\bar{X} = \bar{x}$ is $M(t) = (1-2t)^{-(\frac{n-1}{2})}$ which is independent of $\bar{X}$.

$\therefore$ $\frac{nS^2}{\sigma^2}$ and $\bar{X}$ are stochastically independent and the M.G.F. of $\frac{nS^2}{\sigma^2}$ is $(1-2t)^{-\frac{n-1}{2}}$ which is the M.G.F. of $\chi^2-(n-1)$ distribution.

$\therefore$ $\frac{nS^2}{\sigma^2}$ is $\chi^2-(n-1)$.

since $\frac{ns^2}{\sigma^2}$ and $\bar{X}$ are stochastically independent, S^2 and $\bar{X}$ are also stochastically independent.

Moment Generating Function Technique

Let X_1 and X_2 be stochastically independent with normal distribution $n(\mu_1,\sigma_1^2), n(\mu_2,\sigma_2^2)$. Define the random variable, $Y = X_1 - X_2$ find the P.D.F. of Y.

Solution:

Let$M_Y(t)$ be the M.G.F. of Y.

$M_Y(t) = E[e^{tY}] = E[e^{tX_1-tX_2}]$

$= E[e^{tX_1}e^{tX_2}] = E[e^{tX_1}]E[e^{tX_2}]$

$E[e^{tX_1}] = e^{\mu_1 t+\frac{1}{2}\sigma_1^2t^2}$ (X_1, X_2 are stochastically independent)

E[e$^{-tX_2}$] $= e^{-\mu_2 t+\frac{1}{2}\sigma_2^2t^2}$

$\therefore M_Y(t) = e^{(\mu_1-\mu_2)t+\frac{1}{2}(\sigma_1^2+\sigma_2^2)t^2}$

This is the M.G.F. of normal distribution with mean $\mu_1 - \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$.

$\therefore Y$ is $n(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$

Then the P.D.F. of Y is,

$g(Y) = \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}} e^{-\frac{Y(\mu_1-\mu_2)^2}{2(\sigma_1^2+\sigma_2^2)}}, -\infty < Y < \infty.$

Theorem:

Let $X_1, X_2, ..., X_n$ be mutually stochastically independent random variable having the normal distribution $N(\mu_1, \sigma_1^2), N(\mu_2, \sigma^2)$.......... ...., $N(\mu_n, \sigma_n^2)$. Then the random variable $Y = K_1X_1 + K_2X_2 + ... + K_nX_n$, for real $K_i's$ is normal with mean $\sum_{i=1}^n K_i\mu_i$ and variance $\sum_{i=1}^n K_i^2\sigma_i^2$. $(i.e), Y \sim (\sum_{i=1}^n K_i\mu_i, \sum_{i=1}^n K_i^2\sigma_i^2)$.

Proof:

Let $M_Y(t)$ be the M.G.F. of Y.

The joint P.D.F. of $X_1, X_2, ..., X_n$ is $f \prod_{i=1}^n f_i(x_i)$ where

$f_i(x_i) = \frac{1}{\sqrt{2\pi}\sigma_i} e^{\frac{-(x_i-\mu_i)^2}{2\sigma_i^2}}, -\infty < x < \infty.$

Now, $M_Y(t) = E[e^{ty}] = E[e^{t\sum_{i=1}^n K_iX_i}] = E[\prod_{i=1}^n e^{tK_iX_i}]$

since, $X_1, X_2, ..., X_n$ are mutually stochastically independent.

$M_Y(t) = \prod_{i=1}^n E[e^{tK_iX_i}]$

$= e^{\mu_i(tK_i)+\frac{1}{2}\sigma_i^2(tK_i)^2}$

$M_Y(t) = \prod_{i=1}^{n} e^{K_i\mu_i t+\frac{1}{2}K_i\sigma_i^2 t^2}$

$= e^{\sum K_i\mu_i t+\frac{1}{2}(\sum K_i^2\sigma_i^2)t^2}$

which is the M.G.F. of normal distribution with mean $\sum K_i\mu_i$ and $\sum K_i\sigma_i^2$.

Note:

If $K_i = i \ \forall i$ we say that sum of n stochastically independent normal random variable is again normal with mean the sum of means and variance the sum of the variance.

Theorem:

Let $X_1, X_2, ..., X_n$ be n-mutually stochastically independent random variable respectively having the distribution $\chi^2(r_1), \chi^2(r_2)...., \chi^2(r_n)$ has Chi-square distribution with degree of freedom $r = r_1 + r_2 + ... + r_n$.

Proof:

Let $M_Y(t)$ be M.G.F. of Y.

$M_Y(t) = E[e^{tY}] = E[e^t \sum_{i=1}^{n} X_i = E[\prod_{i=1}^{n} e^{tX_i}]$

$M_Y(t) = \prod_{i=1}^{n} E[e^{tX_i}]$

since $X_1, X_2, ..., X_n$ are mutually stochastically $X_i \sim \chi^2(r_i)$.

$E[e^{tX-i}] = (1-2t)^{-\frac{r_i}{2}}, t < \frac{1}{2}$.

$M_Y(t) = \prod_{i=1}^{n}(1-2t)^{-\frac{r_i}{2}}$

$= (1-2t)^{-\sum_{i=1}^{n}\frac{r_i}{2}}$

$= (1-2t)^{-\frac{r}{2}}$, where $r = \sum_{i=1}^{n} r_i$.

This is M.G.F. of $\chi^2(r)$.

$\therefore$ Y is $\chi^2(r)$ where $r = r_1 + r_2 + ... + r_n$.

Theorem:

Let $X_1, X_2, ..., X_n$ be n mutually stochastically independent random variable respectively having the distribution $n(\mu_1, \sigma_1^2), n(\mu_2, \sigma_2^2), ...$ $....., n(\mu_n, \sigma_n^n)$. Then $Y = \sum_{i=1}^{n}(\frac{X_i-\mu_i}{\sigma_i})^2$ is $\chi^2(n)$.

Proof:

We know that X_i is $n(\mu_i, \sigma_i)^2$ then $(\frac{X_i-\mu_i}{\sigma_i})^2$ is $\chi^2(1)$.

By above theorem, $Y = \sum_{i==1}^{n}(\frac{X_i-\mu_i}{\sigma_i})^2$ is $\chi^2(n)$.

The central limit theorem:

Let $X_1, X_2, ..., X_n$ denote the items of the random sample from a distribution that has mean μ and positive variance σ^2. Then the random variable $Y_n = \frac{\sum_{i=1}^{n}(X_i - n\mu)}{\sigma\sqrt{n}} = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma}$ has a limiting distribution that is normal with mean zero and variance 1.

Proof:

We assume that the M.G.F. of the random sample exists

$-h < t < h$.

Let $M(t) = E[e^t x]; -h < t < h$ and let M(t) denote the M.G.F. of $X - \mu$.

$M(t) = E[e^{t(X-\mu)}] = E[e^{tX}e^{-\mu t}] = e^{-\mu t}E[e^{tX}]$.

Also exists $-h < t < h$.

since, M(t) is the M.G.F. of $X - \mu$, $M(0) = 1$

$M^{'}(0) = E[X - \mu] = 0$

$M^{''}(0) = E[(x - \mu)^2] = \sigma^2$

By Taylor's formula there exist a number $\mathcal{C}$ between 0 and t such that $M(t) = M(0) + M^{'}(0)t + \frac{M^{''}(0)t^2}{2} = 1 + \frac{M^{''}(\mathcal{C})t^2}{2}$

If $\frac{\sigma^2 t^2}{2}$ is add and subtracted, then

$M(t) = 1 + \frac{\sigma^2 t^2}{2} + \frac{M^{''}(t)t^2}{2} - \frac{\sigma^2 t^2}{2}$.

$\therefore M(t) = 1 + \frac{\sigma^2 t^2}{2} + \frac{(M^{''}(t)-\sigma^2)t^2}{2} \longrightarrow (*)$

consider $M(t, n)$,

$M(t, n) = E[e^{tY_n}] = e[e^{t(\frac{\sum_{i=1}^{n}(X_i - n\mu)}{\sigma\sqrt{n}})}]$

$= E[e^{t[(\frac{X_1-\mu}{\sigma\sqrt{n}})+(\frac{X_2-\mu}{\sigma\sqrt{n}})+......+(\frac{X_n-\mu}{\sigma\sqrt{n}})]}]$

$= E[e^{t(\frac{X_1-\mu}{\sigma\sqrt{n}})}e^{t(\frac{X_2-\mu}{\sigma\sqrt{n}})}.......e^{t(\frac{X_n-\mu}{\sigma n})}]$

$= E[\prod_{i=1}^{n} e^{t\frac{(X_i-\mu)}{\sigma\sqrt{n}}}]$

$= \prod_{i=1}^{n}[e^{\frac{t}{\sigma\sqrt{n}}(X_i-\mu)}]$

$= [M(\frac{t}{\sigma\sqrt{n}})]^n, -h < \frac{t}{\sigma\sqrt{n}} < h \longrightarrow (1).$

Replacing t by $\frac{t}{\sigma\sqrt{n}}$ in M(t) in $(*)$, we have,

$M(\frac{t}{\sigma\sqrt{n}}) = 1 + \frac{\sigma^2(\frac{t}{\sigma\sqrt{n}})^2}{2} + \frac{(M^{''}(\mathcal{C})-\sigma^2)(\frac{t}{\sigma\sqrt{n}})^2}{2}$

$= 1 + \frac{\sigma^2 t^2}{2n} + \frac{(M^{''}(\mathcal{C})-\sigma^2)t^2}{2n\sigma^2}$

$(1) \Rightarrow M(t,n) = [1 + \frac{t^2}{2n} + \frac{(M^{''}(\mathcal{C})-\sigma^2)t^2}{2n\sigma^2}]^n, 0 < \mathcal{C} < \frac{t}{\sigma\sqrt{n}}$, as $n \to \infty$, $C \to 0$.

since, $M^{''}(\mathcal{C})$ is continuous at $\mathcal{C} = 0$.

$\lim_{\mathcal{C}\to 0} M^{''}(\mathcal{C}) = M^{''}(0) = 0$

$\lim_{\mathcal{C}\to 0}(M^{''} - \sigma^2) = \sigma^2 - \sigma^2 = 0$

$lim_{\mathcal{C}\to 0} M(t,n) = e^{\frac{t^2}{2}}$ (since, $\lim_{n\to\infty}(1 + \frac{b}{n} + \frac{\Psi(n)C_n}{n})) = e^{bc}$ if $\lim_{n\to\infty}\Psi(n) = 0)$

$\therefore e^{\frac{t^2}{2}}$ is the M.G.F. of n(0, 1).

This Y_n has the limiting normal distribution with mean 0 and variance 1.

Problem:

Let $\bar{X}$ denote the mean of the random sample of size 75 from the distribution that has P.D.F. $f(x) = \begin{cases} 1, & 0 < x < 1. \\ 0, & elsewhere. \end{cases}$

Approximate $P_r(0.45 < \bar{X} < 0.55)$.

Solution:

Let μ be the mean of uniform distribution on the interval (0, 1).

$\mu = E[X] = \int_0^1 x dx = (\frac{x^2}{2})_0^1 = \frac{1}{2}$

$E[X^2] = \int_0^1 x^2 dx = (\frac{x^3}{3})_0^1 = \frac{1}{3}$

$\sigma^2 = E[X^2] - (E[X])^2 = \frac{1}{3} - (\frac{1}{2})^2$

$= \frac{1}{3} - \frac{1}{4} = \frac{4-3}{12} = \frac{1}{12}$

$\sigma^2 = \frac{1}{12} = \frac{1}{4.3} \Rightarrow \sigma = \frac{1}{2\sqrt{3}}$

$P_r(0.45 < \bar{X} < 0.55) = P_r(\frac{\sqrt{n}(0.45-\mu)}{\sigma}) < \frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} < \frac{\sqrt{n}(0.55-\mu)}{\sigma})$.

By central limit theorem, the limiting distribution of $\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma}$ is n(0, 1).

$\therefore$ The required probability, $P_r(\frac{\sqrt{75}(0.45-0.55)}{\frac{1}{2\sqrt{3}}} < \frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} < \frac{\sqrt{75}(0.55-0.5)}{\frac{1}{2\sqrt{3}}})$

$\Rightarrow P_r(-1.5 < \frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} < 1.5)$

$\Rightarrow N(1.5) - N(-1.5) = N(1.5) - 1 + N(1.5)$

$= 2N(1.5) - 1 = 2(0.933) - 1 = 0.866.$

Problem:

Let Y be the $n(100, \frac{1}{2})$ approximate,

$P_r(48, 49, 50, 51, 52) = P_r(47.5 < Y < 52)$

Solution:

Mean $\mu = np = 100.\frac{1}{2} = 50$

Variance $\sigma^2 = npq = np(1-p) = 1000.\frac{1}{2}.\frac{1}{2} = 25$

$\sigma = \pm 5.$

$P_r(47.5 < Y < 52.5) = P_r(\frac{47.5-\mu}{\sigma} < \frac{Y-\mu}{\sigma} < \frac{52.5-\mu}{\sigma}).$

$= P_r(\frac{47.5-50}{5} < \frac{Y-\mu}{\sigma} < \frac{52.5-50}{5})$

$= P_r(\frac{-2.5}{5} < \frac{Y-\mu}{\sigma} < \frac{2.5}{5})$

$= P_r(-0.5 < \frac{y-\mu}{\sigma} < 0.5)$

But $\frac{Y-\mu}{\sigma}$ has limiting n(0, 1).

$P_r(47.5 < Y < 52.5) = N(0.5) - N(0.5)$

$= N(0.5) - 1 + N(0.5) = 2N(0.5) - 1$

$= 2(0.691) - 1 = 0.382$

$P_r(47.5 < Y < 52.5) = 0.382.$

Problem:

$n(\mu, 12)$ Find $P_r(2.30 < S^2 < 22.2)$.

Solution:

Let S^2 be the variance of a sample of size from the normal distribution.

To find: $P_r(2.30 < S^2 < 22.2)$

we know that $\frac{nS^2}{\sigma^2} \sim X^2(n-1)$.

$P_r(2.30 < S^2 < 22.2) = P_r(\frac{2.30n}{\sigma^2} < \frac{nS^2}{\sigma^2} < \frac{2.2n}{\sigma^2})$

$= P_r(\frac{2.30\times 6}{12} < \frac{nS^2}{\sigma^2} < \frac{22.2\times 6}{12})$

$= P_r(1.15 < \frac{nS^2}{\sigma^2} < 11.1)$

$\gamma = n - 1 = 6 - 1 = 5.$

Problem:

Find the P.D.F. of the sample variance S^2 provided that the distribution from the sample is $n(\mu, \sigma^2)$.

Solution:

Now $S^2 = \frac{\sigma^2}{n}(\frac{nS^2}{\sigma^2})$.

$S^2 = \frac{\sigma^2}{n}Y$, where $Y = \frac{nS^2}{\sigma^2}$.

M.G.F. of S^2 is $E[e^{ts^2}] = E[e^{t\frac{\sigma^2}{n}Y}] = [e^{\mu t\frac{\sigma^2}{n} + \frac{\sigma^4 t^2}{2n^2}}] = e^{\frac{\mu t\sigma^2}{n} + \frac{\sigma^4 t^2}{2n^2}}$

(i.e), M.G.F. of S^2 is $e^{\frac{\mu t\sigma^2}{n} + \frac{t^2\sigma^4}{2n^2}}$

with mean $\frac{\mu\sigma^2}{n}$ and variance $\frac{\sigma^4}{n^2}$.

The P.D.F. of $S^2 = \frac{1}{\sqrt{2\pi}(\frac{\sigma^2}{n})}e^{\frac{-1}{2}[\frac{\frac{\sigma^2}{n}(y-\mu)}{\frac{\sigma^2}{n}}]^2} = \frac{n}{\sigma^2\sqrt{2\pi}}e^{\frac{-1}{2}(y-\mu)^2}$.

which is a P.D.F of S^2.